تفاعـــــلات حـــــــل صدمـــــــة دلتا المنقســــــمة لنمـــــــــوذج (الكروماتــــــوجـــــــرافيا) الخاص

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Interactions of split delta shocks solution for Simplified chromatography model

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الملخص:

يتـــــم إجـــــــراء صدمة دلتا المنقسمة لتوفـــــير طريقة لاستخدامها في حلّ المعادلات التفاضلية الجزئية ، وقد تم تعريف صدمة دلتا المنقسمة ، وبعد ذلك تم استخدامها لحل مسألة ريمان بالقيم الابتدائية بالإضافة إلى استخدام (الانتروبيا) ، والتفاعلات لحلــــول دلتا والصدمات الفرديــــة لنموذج (الكروماتوجرافيا) الخاص، الحل الذي تم الحصول عليه فريد من نوعه بالمعنى التوزيـــعي.

Abstract

A split delta shock is made to provide a way of using it in nonlinear PDEs. We define a split delta shock and use it for solving Riemann initial data problem with using entropies and interactions for delta and singular shocks solutions for a special chromatography model. The obtained solution is unique in distributional sense.

Introduction:

A split delta shock is a representation of the Dirac delta shock function. The split delta shocks are introduced in order to solve some systems of conservation laws without classical solutions (see [1]). The main idea is to split a physical domain $\Omega \subset RxR_+$ into pieces with using a simpler condition- so-called overcompressibility: All characteristics should run into the shock curve, for the following Riemann problem for simplified chromatography model

$$u_t + \left(\frac{u}{1-u+v}\right)_x = 0, v_t + \left(\frac{v}{1-u+v}\right)_x = 0 \ \rightarrow (1)$$

One can also look in [3] about entropies and interactions for delta and singular shocks solutions. Physical domain for solutions is defined by $1 - u + v > 0$ or $v - u > -1$.

The Physical domain split into pieces. In the interior of each such piece, one has a classical solution to the system while a boundary could support a signed delta measure.

Such a solution is called a split delta shock. The procedure works well if the system is linear in one of the variables.

2- The definition of split delta shocks

Let $\Omega_i \neq \emptyset,\ i = 1, \dots, n$ be a finite family of disjoint open sets with piecewise smooth boundary curves $\Gamma_i, \mathrm{i} = 1, \dots, \mathrm{m}: \Omega_\mathrm{i} \cap \Omega_j = \emptyset,\ \cup_{\mathrm{i}=1}^{\mathrm{n}} \overline{\Omega}_\mathrm{i} = \overline{\mathrm{R}_+^2}$

where $\overline{\Omega}_i$ denotes the closure of Ω_i. Denote by $C(\overline{\Omega}_i)$ the space of bounded and continuous real- valued functions on $\overline{\Omega}_i$, equipped with the L^∞ $-$norm. Let $\mathrm{M}(\overline{\Omega}_i)$, be the space of measures on $\overline{\Omega}_i$.

Define

$C_\Gamma = \prod_{i=1}^{n} C(\overline{\Omega}_i)$ $, M_\Gamma = \prod_{i=1}^{n} M(\overline{\Omega}_i)$.

The multiplication of G=$G_1, \dots G_n \in C_\Gamma and\ D = (D_1, \dots D_n) \in M_\Gamma$ is defined to be an element, $D.G = (D_1 G_1, \dots, D_n G_n) \in M_\Gamma$where each component is defined as the usual product of a continuous function and a measure.

Every measure on $\overline{\Omega}_i$ can be identified with a measure defined on $\overline{R_+^2}$ with support in $\overline{\Omega}_i$. Thus one can define the mapping m in the following way

$$m: M_\Gamma \to M\left(\overline{R_+^2}\right), m(D) = D_1 + D_2 + \cdots + D_n.$$

is divided into two regions Ω_1, Ω_2A typical example is obtained when $\overline{R_+^2}$ by a piecewise smooth curve $x = \gamma(t)$. The delta function along the line $\delta\big(x - \gamma(t)\big)\epsilon\ M\left(\overline{R_+^2}\right)$ x=γ(t) can be split in way a non unique way into a left hand side$D^- \in M(\overline{\Omega}_1)$ and the right – hand component $\mathrm{D}^+ \in M(\overline{\Omega}_2)$*such that*

$$\delta\big(x - \gamma(t)\big) = m(a_0(t)D^- + a_1(t)D^+)$$

with $\alpha_0(t) + \propto_1 (t) = 1$. The solution concept which allows to incorporate such two sided delta functions as well as shock wave is modeled along the lines of the classical weak solution concept and proceeds as follows:
Step1:Perform all nonlinear operations of function in space C_Γ.
Step2:Perform multiplications with measures in the space M_Γ.
Step3:Map the space M_Γ into $\left(\overline{R_+^2}\right)$ by means the map m and embed it into the space of distributions.
Step4: Perform the differentiation in the sense of distributions and require that the equation is satisfied in this sense.

Suppose that there exist a two components split delta shock solution.

$u(x,t) = \begin{cases} u_0\,, & x \leq ct \\ u_1\,, & x \geq ct \end{cases} + (\alpha_0\delta^- + \alpha_1\delta^+)t\,, v(x,t) = \begin{cases} v_0\,, & x \leq ct \\ v_1\,, & x \geq ct \end{cases} + (\beta_0\delta^- + \beta_1\delta^+)t \rightarrow (2),$

defined in [4]and[5] to some conservation law system linear in one of solution component, v for definiteness. A construction of an appropriate entropies and interactions for delta and singular shocks solutions is straightforward; Put

$$\lim_{\varepsilon\to 0} a_\varepsilon\, u_{1,\varepsilon} = \propto_0 \;,\; \lim_{\varepsilon\to 0} b_\varepsilon\, u_{2,\varepsilon} = \propto_1$$

$\lim_{\varepsilon\to 0} a_\varepsilon\, v_{1,\varepsilon} = \beta_0 \;\;, \lim_{\varepsilon\to 0} b_\varepsilon\, v_{2,\varepsilon} = \beta_1.$

3- Elementary waves and overcompressibility condition

In the case $u_0 > v_0$, $u_1 > v_1$, there is no elementary waves solution to the Riemann problem for simplified chromatography model (1), we can try to substitute the entropies and interactions for delta and singular shocks solutions(see[3])

$$u(x,t) = \begin{cases} u_0\,, & x < (c - a_\varepsilon)t \\ u_{1,\varepsilon}\,, & (c - a_\varepsilon)t < x < ct \\ u_{2,\varepsilon}\,, & ct < x < (c - b_\varepsilon)t \\ u_1\,, & x > (c - b_\varepsilon)t \end{cases} \rightarrow (3)$$

in both equations of the system. All an admissibility criteria for the solution we will use the overcompressibility condition.

Definition: The system of the form (3) is called overcompressibility condition if

$\lambda_1(u_0, v_0) \geq c \geq \lambda_2(u_1, v_1)$

i .e. all characteristics should run into curve . One can look in [9] or [10] for a detailed explanation of that admissibility condition.

Theorem 1: *There exists a unique solution to the simplified chromatography model (1) in the region where u ,v and 1 – u + v are non – negative. The solution consists of elementary waves, vacuum states, and satisfies an overcompressibility condition with split delta shock.*

In the sense of distribution the uniqueness holds.

Proof:

The system (1) has the eigenvalues

$\lambda_1(u,v) = \frac{1}{1-u+v}$, $\lambda_2(u,v) = \frac{1}{(1-u+v)^2}$

with the appropriate eigenvectors $r_1 = (1.\ 1)$ and $r_2 = (1, \frac{v}{u})$ where the 1− field is linearly degenerate , while 2- filed is genuinely nonlinear for $v \neq u$.

The states when u or v equals to zero (“ vacuum in u or v ”) .

Let us try with a split delta shock solution of the form (1), with the following initial data

$$u(x,0) = \begin{cases} u_0 & ,x < 0 \\ u_1 & ,x > 0 \end{cases} \quad ,v(x,0) = \begin{cases} v_0 & ,x < 0 \\ v_1 & ,x > 0 \end{cases}$$

with the special function (3).

The form (3) is general enough for solving Riemann problem (1) as are could see below.

Now , suppose that $u_0 > v_0, u_1 < v_1$ and substitute a function of the *form (3) into the system (1) . For the first equation, we have*

$$I_1 \approx -\int_0^\infty \int_{-\infty}^{(c-a_\varepsilon)t} \left(u_0 \partial_t \varphi(x,t) + (\frac{u_0}{1-u_0+v_0}) \partial_x \varphi(x,t)\right) dx\, dt$$

$$-\int_0^\infty \int_{(c-a_\varepsilon)t}^{ct} \left(u_{1,\varepsilon} \partial_t \varphi(x,t) + (\frac{u_{1,\varepsilon}}{1-u_{1,\varepsilon}+v_{1,\varepsilon}}) \partial_x \varphi(x,t)\right) dx\, dt$$

$$-\int_0^\infty \int_{ct}^{(c+b_\varepsilon)t} \left(u_{2,\varepsilon} \partial_x \varphi(x,t) + (\frac{u_{2,\varepsilon}}{1-u_{2,\varepsilon}+v_{2,\varepsilon}}) \partial_x \varphi(x,t)\right) dx\, dt$$

$$-\int_0^\infty \int_{(c+b_\varepsilon)t}^{\infty} \left(u_1, \partial_x \varphi(x,t) + (\frac{u_1}{1-u_1+v_1}) \partial_x \varphi(x,t)\right) dx\, dt$$

where $\varphi \in C_0^\infty(R)$.

After integration by parts and calculating; we get

$$I_1 \approx -\int_0^\infty u_0(c-a_\varepsilon)\, \varphi\big((c-a_\varepsilon)t.t\big)dt - \int_0^\infty \left(\frac{u_0}{1-u_0+v_0}\right) \varphi\big((c-a_\varepsilon)t,t\big)dt$$

$$-\int_0^\infty u_{1,\varepsilon}\, \varphi\big((c-a_\varepsilon)t,t\big)(a_\varepsilon)dt - \int_0^\infty \left(\frac{u_{1,\varepsilon}\, a_\varepsilon}{1-u_{1,\varepsilon}+v_{1,\varepsilon}}\right) \varphi\big((c-a_\varepsilon)t,t\big)dt$$

$+\int_0^\infty u_{2,\varepsilon}\,(b_\varepsilon)\varphi\big((c+b_\varepsilon)t,t\big)dt+\int_0^\infty\left(\frac{u_{2,\varepsilon}\,b_\varepsilon}{1-u_{2,\varepsilon}+v_{2,\varepsilon}}\right)\varphi\big((c+b_\varepsilon)t,t\big)dt$

$+\int_0^\infty u_1\,(c+b_\varepsilon)\,\varphi\big((c+b_\varepsilon)t\,,t\,\big)\big)\,dt+\int_0^\infty\left(\frac{u_1}{1-u_1+v_1}\right)\varphi\big((c+b_\varepsilon)t,t\big)dt$

The sign ,, ≈,, simple means a convergence to zero as $\varepsilon \to 0$.

Note that

$+\int_\infty^0 u_0\,\varphi(x,0)\;\,dx+\;\int_0^\infty u_1\,\varphi(x,0)dx=\langle u|\;_{t=0}\,,\varphi\rangle$

also

$+\int_\infty^0 u_{1,\varepsilon}\,\varphi(x,0)\;\,dx+\;\int_0^\infty u_{2,\varepsilon}\,\varphi(x,0)dx=\langle u|\,_{t=0}\;,\varphi\rangle$

that cancels with initial data and we will drop it in the rest of calculations. since we will use a split delta shock then we put :

$\lim_{\varepsilon\to0} a_\varepsilon\, u_{1,\varepsilon} =\propto_0\;,\;\lim_{\varepsilon\to0} b_\varepsilon\, u_{2,\varepsilon} =\propto_1\;\;\to\;(4)$

and we used the fact

$\varphi((c\pm a_\varepsilon)t,t)=\varphi(c,t)\pm\varphi\partial_t(c,t)a_\varepsilon t+O(\varepsilon^2)$

also

$\varphi((c\pm b_\varepsilon)t,t)=\varphi(c,t)\pm\varphi\partial_t(c,t)b_\varepsilon t+O(\varepsilon^2)$

Then we get the following equation

$-\int_0^\infty\Big(c[u]-\Big[\frac{u}{1-u+v}\Big]+\big(a_\varepsilon u_{1,\varepsilon}+b_\varepsilon u_{2,\varepsilon}\big)a_\varepsilon\varphi(ct,t)\Big)\,dt$

$-\int_0^\infty c\big(a_\varepsilon u_{1,\varepsilon}+b_\varepsilon u_{2,\varepsilon}\big)+\left(\frac{u_{1,\varepsilon}\,a_\varepsilon}{1-u_{1,\varepsilon}+v_{1,\varepsilon}}\right)+\left(\frac{u_{2,\varepsilon}\,b_\varepsilon}{1-u_{2,\varepsilon}+v_{2,\varepsilon}}\right)\partial_x\varphi(ct,t)dt=0$

In the sequel, the notation [u] means u_1-u_0 and $\Big[\frac{u}{1-u+v}\Big]$ *means* $\frac{u_1}{1-u_1+v_1}-\frac{u_0}{1-u_0+v_0}$. The above relation is true if and only if

$\lim_{\varepsilon\to0}\big(a_\varepsilon\,u_{1,\varepsilon}+b_\varepsilon u_{2,\varepsilon}\big)=k_1=c[u]-[\frac{u}{1-u+v}]$

$\propto_0+\propto_1=k_1=c[u]-[\frac{u}{1-u+v}]\to(5)$

$\lim_{\varepsilon\to0} c\big(a_\varepsilon\,u_{1,\varepsilon}+b_\varepsilon u_{2,\varepsilon}\big)=ck_1=\frac{\lim_{\varepsilon\to0}\;a_\varepsilon u_{1,\varepsilon}}{1-u_{1,\varepsilon}+v_{1,\varepsilon}}-\frac{\lim_{\varepsilon\to0}\;b_\varepsilon u_{2,\varepsilon}}{1-u_{2,\varepsilon}+v_{2,\varepsilon}}$

$c\,k_1=c(\propto_0+\propto_1)=\frac{\propto_0}{1+A_0}+\frac{\propto_1}{1+A_1}\to(6)$

We used fact $v_{1,\varepsilon}-u_{1,\varepsilon}\;\to A_0\;<\;\infty$

and $v_{2,\varepsilon}-u_{2,\varepsilon}\;\to A_{\,1}<\;\infty$

otherwise, $k_{1,2}$ would be zero.

With the same method, and with substitution

$u_t \to v_t$, $\frac{u}{1-u+v} \to \frac{v}{1-u+v}$

For the second equation

$v_t + \left(\frac{v}{1-u+v}\right)_x = 0$

Since we using a spilt delta shock, then now we will put:

$\lim_{\varepsilon\to 0} a_\varepsilon\, v_{1,\varepsilon} = \beta_0$, $\lim_{\varepsilon\to 0} b_\varepsilon\, v_{2,\varepsilon} = \beta_1 \to (7)$

Then we get the following equation:

$$\int_0^\infty \left(c[v] - \left[\frac{v}{1-u+v}\right] + \left(a_\varepsilon v_{1,\varepsilon} + b_\varepsilon v_{2,\varepsilon}\right) a_\varepsilon \varphi(ct,t)\right) dt$$

$$-\int_0^\infty \left(c\left(a_\varepsilon v_{1,\varepsilon} + b_\varepsilon v_{2,\varepsilon}\right) + \left(\frac{a_\varepsilon\, v_{1,\varepsilon}}{1-u_{1,\varepsilon}+v_{1,\varepsilon}} + \frac{b_\varepsilon\, v_{2,\varepsilon}}{1-u_{2,\varepsilon}+v_{2,\varepsilon}}\right) a_\varepsilon\, t\partial_x(ct,t)\right) dt = 0$$

Then

$\lim_{\varepsilon\to 0}\left(a_\varepsilon\, v_{1,\varepsilon} + b_\varepsilon v_{2,\varepsilon}\right) = k_2 = c[v] - \left[\frac{v}{1-u+v}\right]$

$\beta_0 + \beta_1 = k_2 = c[v] - \left[\frac{v}{1-u+v}\right] \to (8)$

and

$\lim_{\varepsilon\to 0} c\left(a_\varepsilon\, v_{1,\varepsilon} + b_\varepsilon v_{2,\varepsilon}\right) = \frac{\lim_{\varepsilon\to 0} a_\varepsilon\, v_{1,\varepsilon}}{1+A_0} + \frac{\lim_{\varepsilon\to 0} b_\varepsilon\, v_{2,\varepsilon}}{1+A_1}$

$c(\beta_0 + \beta_1) = ck_2 = \frac{\beta_0}{1+A_0} + \frac{\beta_1}{1+A_1} \to (9)$

Also the notation here [v] means v_1-v_0 and $\left[\frac{v}{1-u+v}\right]$ means $\frac{v_1}{1-u_1+v_1} - \frac{v_0}{1-u_0+v_0}$

But sometimes $\lambda_1 > \lambda_2$ the over compressibility (when $\lambda_1 > 1$).

Then, we need both

$\frac{1}{1-u_0+u_0} \geq c \geq \frac{1}{(1-u_1+v_1)}$

and

$\frac{1}{(1-u_0+v_0)^2} \geq c \geq \frac{1}{(1-u_1+v_1)^2}$

If $A_i < \infty$, *then* $\propto_i = \beta_i$, *and if* $A_i = \infty$ *then* $\beta_i > \propto_i$

$\propto_0 = \beta_0$, $\propto_1 = \beta_1$ *And* $A_i = \infty$. *Then* **Case I**: *Let* $A_0 < \infty$

So $k_1 = k_2$

Then from (5) we have

$ck_1 - \frac{\alpha_0}{1+A_0} - \frac{\alpha_1}{1+A_1} = 0 \to (10)$

is the same from (9) we have

$\rightarrow$ (11) = 0ck$_2$ $-\frac{\beta_0}{1+A_0} - \frac{\beta_1}{1+A_1}$

So will use only (10).

Since $k_1 = k_2$, we conclude that

$c[u] - \left[\frac{u}{1-u+v}\right] = c\,[v] - \left[\frac{v}{1-u+v}\right]$

Then

$c\,(v_1 - v_0 - u_1 + u_0) = \frac{v_1-u_1}{1-u_1+v_1} + \frac{v_0-u_0}{1-u_0+v_0}$

After calculating, we get

$c = \frac{1}{(1-u_1+v_1)(1-u_0+v_0)}$

Let us now check the admissibility condition:

$u_0 , v_0) \geq c \geq \lambda_1 . (u_1 , v_1)(\lambda_1$

$u_0 , v_0) \geq c \geq \lambda_2 \ (u_1 , v_1) \ \ (\lambda_2$

1-If

$\frac{1}{1-u_0+v_0} \geq \frac{1}{(1-u_1+v_1)(1-u_0+v_0)}$ $then\ 1 \geq \frac{1}{1-u_1+v_1}$

2- If

$$\frac{1}{(1-u_0+v_0)^2} \geq \frac{1}{(1-u_1+v_1)(1-u_0+v_0)}$$

then

$$\frac{1}{1-u_0+v_0} \geq \frac{1}{(1-u_1+v_1)^2}$$

3- If

$$\frac{1}{(1-u_1+v_1)(1-u_0+v_0)} \geq \frac{1}{1-u_1+v_1} \quad then \quad \frac{1}{1-u_0+v_0} \geq 1$$

4-If

$$\frac{1}{(1-u_1+v_1)(1-u_0+v_0)} \geq \frac{1}{(1-u_1+v_1)^2}$$

then

$$\frac{1}{1-u_0+v_0} \geq \frac{1}{1-u_1+v_1}$$

Thus , The overcompressive condition is satisfied if

$$\frac{1}{1-u_0+v_0} \geq 1 \geq \frac{1}{1-u_1+v_1}.$$

Case 2 : Let one of A_1 , A_2 is ∞ (that is $v_\varepsilon - u_\varepsilon \to \infty$), say $A_2 = \infty$. Then $\beta_0 = \propto_0$, $\beta_1 > \propto_1 \Rightarrow k_1 < k_2$

From (10) we have

$$ck_1 - \frac{\propto_0}{A_0} = 0$$

and from (11), we have

$$ck_2 - \frac{\beta_0}{A_0} = 0$$

since $\propto_0 = \beta_0$, it follows that $k_1 = k_2$ and that is a contradiction .

Thus only **Case I** is possible.

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