استخدام تحويل عبعوب - شخيم لحل المعادلات التفاضلية الجزئية

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لقد قمنا باشتقاق تحويل عبعوب- شخيم للمشتقه الجزئيه ، وتم اثبات قابليته للتطبيق باستخدام أربع معادلات جزئية مختلفة. في هذا البحث نجد الحلول الخاصة لهذه المعادلات.

The Use of Abaoub- Shkheam Transform for Solving Partial Differential Equations

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Abstract

Abaoub- Shkheam Transform of partial derivative is derived, and its applicability demonstrated using four different partial equations. In this paper we find the particular solutions of these equations.

Keywords: Abaoub- Shkheam Transform- Partial Differential Equations.

1-introdaction:

For a long time, differential equations have played a central role in all aspects of applied mathematics, and their importance has grown with the advent of the computer. Thus, the investigation and analysis of differential equations cruising in applications resulted in many deep mathematical problems; thus, there are numerous techniques for solving differential equations. The integral transform was widely used, and thus several words on the theory and applications of integral transforms have been coined, including Laplace, Fourier, Mellin, Hankel, and Sumudu, to name a few. Ali. Abaoub, and Abigail. Shkheam Abaoub- Shkheam recently introduced a new integral transform, dubbed the Abaoub- Shkheam transform, and used it to solve ordinary and partial differential equations. Our purpose here is to show the applicability of this interesting new transformation and its effect on solving such problems.

2- Definition and Derivations the Abaoub- Shkheam Transform of Derivatives.

Definition1.2: Let $f(t)$ be a function defined for all $t \geq 0$, the Q-transform of $f(t)$ is the function $T(u,s)$ defined by $T(u,s) = Q[f] =$

$$\int_0^\infty f(ut)\, e^{-\frac{t}{s}}\, dt \qquad (1)$$

provided the integral exists for some s, where $s \in (-t_1, t_2)$.

The original function $f(t)$ in (1) is called the inverse transform or inverse of $T(u,s)$, and is denoted by $f(t) = Q^{-1}\{T(u,s)\}$.

If we substitute $ut = y$, then equation (1) becomes,

$$Q[f(t)] = T(u,s) = \frac{1}{u}\int_0^\infty f(y)\, e^{-\frac{1}{us}y}\, dy \qquad (2)$$

To obtain Abaoub- Shkheam transform of partial derivatives we use integration by parts as follows:

$$Q\left[\frac{\partial f(x,t)}{\partial t}\right] = \int_0^\infty \frac{\partial f(x,t,u)}{\partial t} e^{\frac{-t}{s}}\, dt$$

$$\lim_{p\to\infty}\int_0^p \frac{\partial f(x,t,u)}{\partial t} e^{\frac{-t}{s}}\, dt = \lim_{p\to\infty} f(x,t)e^{\frac{-t}{s}} \Big|_0^p + \frac{1}{s}\int_0^p e^{\frac{-t}{s}} f(x,t,u)dt$$

$$= -\frac{1}{u}v(x,0) + \frac{1}{us}V(x,s,u)$$

Thus $Q\left[\frac{\partial f(x,t)}{\partial t}\right] = -\frac{1}{u}v(x,0) + \frac{1}{us}V(x,s,u) \qquad (3)$

To find $Q\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right]$, let $\frac{\partial f(x,t)}{\partial t} = g(x,t)$, then by using Eq. (3) we have:

$$Q\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = Q\left[\frac{\partial g(x,t)}{\partial t}\right] = Q[g(x,t)] - g(x,t)$$

$$Q\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right]=\frac{V(x,s,u)}{u^2s^2}-\frac{1}{u^2s}v(x,0)-\frac{\partial v}{\partial t}(x,0) \qquad (4)$$

We can easily extend this result to the nth partial derivative by using mathematical induction.

Now, we assume the $f(x,t)$ is piecewise continuous and is of exponential order. Then,

$$Q\left[\frac{\partial f(x,t)}{\partial t}\right]=\int_0^\infty \frac{\partial f(x,t,u)}{\partial t} e^{\frac{-t}{s}}\,dt$$

Using the Leibniz' rule $Q\left[\frac{\partial f(x,t)}{\partial t}\right]=\frac{\partial}{\partial x}\int_0^\infty e^{\frac{-t}{s}} f(x,u,t)dt$

Thus $Q\left[\frac{\partial f(x,t)}{\partial t}\right]=\frac{d}{dx}V(x,u,s)$ (5)

Also we can find $Q\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right]=\frac{d^2}{dx^2}V(x,u,s)$ (6)

In summary $Q\left[\frac{\partial^n f(x,t)}{\partial x^n}\right]=\frac{d^n}{dx^n}V(x,u,s)$ (7)

3 -Partial Differential Equations.

Now to illustrate the method we consider the general linear partial integral differential equation

$$\sum_{i=0}^{m} a_i \frac{\partial^i u}{\partial t^i}+\sum_{i=0}^{n} b_i \frac{\partial^i u}{\partial x^i}+cu+f(x,t)=0$$

Applying a **Abaoub- Shkheam** transformation to Equation 2, we get

$$\sum_{i=0}^{m} a_i\, Q\left[\frac{\partial^i u}{\partial t^i}\right]+\sum_{i=0}^{n} b_i\, Q\left[\frac{\partial^i u}{\partial x^i}\right]+cQ[u]+Q[f(x,t)]=0$$

Using (3)and (4), we get

$$\sum_{i=0}^{m} a_i \frac{\mathrm{V(x,u,s)}}{u^i s^i} - \frac{1}{u}\sum_{k=0}^{i-1} \frac{v^k(\mathrm{x},0)}{(us)^{i-k-1}} + \sum_{i=0}^{n} b_i \frac{d^i \mathrm{V(x,u,s)}}{dx^i} + c\mathrm{V}(x,u,s)$$
$$+ \bar{\mathrm{f}}(x,0) = 0$$

Where $\mathrm{V(x,u,s)} = Q[u(x,t\ )]$ $,\bar{\mathrm{f}}(\mathrm{x,u,s}) = Q[f(x,t)$.

4-Solution of Partial Differential Equations

In this section, we solve first- order partial differential equations and second- order partial differential equations, as well as wave equations, heat equations, Laplace's equations, and Telegraphers equations. Fundamental equations are found in many branches of mathematics. in physics, applied mathematics, and engineering.

Example 4.1.

Find the solution of the first order initial value problem:

$u_x(x,t) - 2u_t(x,t) = u(x,t), x > 0, t > 0,$ (8)

With the initial conditions

$u(x,0) = e^{-3x}$ $u(0,t) =) e^{-2t}$

Taking Abaoub- Shkheam transform of Eq. (8), we have

$$V'(x,u,s) - \frac{2}{\mathrm{us}} V(x,s,u) + \frac{2}{\mathrm{u}} v(x,0) = V(x,u,s)$$

Where $V(x,u,s)$ is Abaoub- Shkheam transform of $u(x,t)$.

By applying the initial condition, we get

$$V'(x,u,s) - \left[\frac{2}{\mathrm{us}} + 1\right] V(x,s,u) = -\frac{2}{\mathrm{u}} e^{-3x}$$

This is the linear ordinary differential equation, it has the integration factor

$$F = e^{-\int(\frac{2}{us}+1)dx} = e^{-(\frac{2}{us}+1)x}$$

Therefore, $V(x,s,u) = \frac{\mathrm{s}}{2\mathrm{us}+1} e^{-3x}$+c (9)

Now $\mathrm{Q[u}(0,t)] = V(0,u,s\ \ ) = Q[e^{-2t}] = \frac{\mathrm{s}}{2\mathrm{us}+1}$ (10)

Compare (10) in (9) , we get c = 0

Applying inverse Abaoub- Shkheam transform on both sides

$$V(x,s,u)=\frac{\mathrm{s}}{2\mathrm{us}+1}e^{-3x}$$

$$u(x,s,u)=Q^{-1}[V(x,s,u)]=Q^{-1}\left[\frac{\mathrm{s}}{2\mathrm{us}+1}\right]e^{-3x}=e^{-2t}e^{-3x}$$

$$u(x,s,u)=e^{-2t}e^{-3x}$$

Example 4.2.

Let's consider the wave equation :

$u_{tt}-u_{xx}=0$ $\quad ,0\le x\le \pi$ $\quad ,t\ge 0$ (11)

With the initial conditions:

$u(0,t)=0\ ,u_t(x,0)=0\ ,u(\pi,t)=0\ ,\ u(x,0)=sinx$

Taking Abaoub- Shkheam transform of Eq. (11), we have

$$\frac{V(x,s,u)}{u^2s^2}-\frac{1}{u^2\mathrm{s}}v(x,0)-\frac{\partial v}{\partial t}(x,0)-V''(\mathrm{x},\mathrm{u},\mathrm{s})=0$$

$$\frac{V(x,s,u)}{u^2s^2}-\frac{1}{u^2\mathrm{s}}sinx-V''(\mathrm{x},\mathrm{u},\mathrm{s})=0$$

$$V''(\mathrm{x},\mathrm{u},\mathrm{s})-\frac{1}{u^2s^2}V(x,s,u)=-\frac{1}{u^2\mathrm{s}}sinx$$

$$V(x,s,u)=c_1e^{\frac{1}{us}x}+c_2e^{\frac{1}{us}x}-\frac{\mathrm{s}}{u^2s^2+1}\sin x \qquad (12)$$

Now $\quad Q[u\ (0,t)]=V\ (0,u,s)=0$ (13)

$\quad \mathrm{Q}[\mathrm{u}(\pi,t)]=V(\pi,u,s\ \ )=0$ (14)

Using(13) and (14)in (12) we get,

$c_1+c_2=0$ (15)

And $\quad c_1e^{\frac{1}{us}\pi}+c_2e^{-\frac{1}{us}\pi}=0$ (16)

Solving(15) and (16) we get,

$$c_1=c_2=0$$

$$V(x,u,s)=-\frac{\mathrm{s}}{u^2s^2+1}sinx \qquad (17)$$

Taking inverse Abaoub- Shkheam transform in(17) we get,

$$u(x,t)=\cos t\sin x$$

Example 4. 3.

Let's consider the homogeneous heat equation in one dimension in a normalized form:

$$u_t = u_{xx}\ ,\ \ u(x,0) = sin\frac{\pi}{l}x\ ,\ \ \ u(0,t) = u(l,t) = 0 \qquad (18)$$

With the initial conditions**:**

$$u_t(x,0) = 0\ \ ,u(\pi,t) = 0$$

Taking Abaoub- Shkheam transform of Eq. (18), we have

$$\frac{1}{\mathrm{us}}V(x,s,u) - \frac{1}{\mathrm{u}}v(x,0) = V''(x,u,s)$$

$$V''(x,s,u) = \frac{1}{\mathrm{us}}V(x,s,u) - \frac{1}{\mathrm{u}}sin\frac{\pi}{l}x$$

$$V''(x,s,u) - \frac{1}{\mathrm{us}}V(x,s,u) = -\frac{1}{\mathrm{u}}sin\frac{\pi}{l}x$$

$$V(x,s,u) = c_1e^{\sqrt{\frac{1}{us}}x} + c_2e^{-\sqrt{\frac{1}{us}}x} + \frac{\boldsymbol{l^2s}}{\boldsymbol{us\pi^2+l^2}}\sin\frac{\pi}{l}x \qquad (19)$$

Now $\quad Q[u\ (0,t)] = V\ (0,u,s) = 0 \qquad (20)$

$\mathrm{Q}[\mathrm{u}(l,t)] = V(l,u,s\ \ \ ) = 0 \qquad (21)$

Using(20) and (21)in (19) we get,

$$c_1 = c_2 = 0$$

$$V(x,s,u) = -\frac{sl^2}{us\pi^2+l^2}\sin\frac{\pi}{l}x \qquad (22)$$

Taking inverse Abaoub- Shkheam transform in(22) we get,

$$u(x,t) = e^{\frac{-\pi^2}{l^2}t}\sin\frac{\pi}{l}x.$$

Example 4.4.

Let's consider the homogeneous Laplace equation:

$u_{xx} + u_{tt} = 0,\ u(x,0) = 0, u_t(x,0) = cosx\ , x, t > 0 \qquad (23)$

Taking Abaoub- Shkheam transform of Eq. (23), we have

$$V''(x,s,u)+\frac{V(x,s,u)}{u^2s^2}-\frac{1}{u^2\mathrm{s}}v(x,0)-\frac{\partial v}{\partial t}(x,0)=0$$

$$V''(x,s,u)+\frac{V(x,s,u)}{u^2s^2}=cosx$$

This is the second-order ordinary differential equation have the particular solution in the form:

$$V(x,s,u)=\frac{cosx}{\mathrm{D}^2+\frac{1}{\mathrm{u}^2\mathrm{s}^2}}=\frac{cosx}{-1+\frac{1}{\mathrm{u}^2\mathrm{s}^2}}=\frac{\mathrm{u}^2\mathrm{s}^2cosx}{-1+\mathrm{u}^2\mathrm{s}^2}=\frac{\mathrm{u}^2\mathrm{s}^2cosx}{\mathrm{u}^2\mathrm{s}^2-1} \qquad (24)$$

Taking inverse Abaoub- Shkheam transform in(24) we get

$$u(x,t)=\sinh t\cos x.$$

Example 4.5.

Let's consider the telegraphers equation:

$$\mathrm{u_{tt}(x,t)+2\alpha u_t(x,t)=\alpha^2u_{xx}(x,)} \qquad ,0<\mathrm{x}<1 \ \ .\mathrm{t}>0 \qquad (25)$$

With the initial conditions:

$$u(x,0)=cosx,\quad u_t(x,0)=0$$

Taking Abaoub- Shkheam transform of Eq. (25), we have

$$\frac{V(x,s,u)}{u^2s^2}-\frac{1}{u^2\mathrm{s}}v(x,0)-\frac{\partial v}{\partial t}(x,0)+2\alpha[\frac{1}{\mathrm{us}}V(x,s,u)-\frac{1}{\mathrm{u}}v(x,0)]$$
$$=\alpha^2V''(x,s,u)$$

$$\alpha^2V''(x,s,u)=\frac{V(x,s,u)}{u^2s^2}+2\alpha\frac{1}{\mathrm{us}}V(x,s,u)-[\frac{1}{u^2\mathrm{s}}cosx+\frac{2\alpha}{\mathrm{u}}cosx]$$

$$\alpha^2V''(x,s,u)-\left[\frac{1}{u^2s^2}+2\alpha\frac{1}{\mathrm{us}}\right]V(x,s,u)=-[\frac{1}{u^2\mathrm{s}}+\frac{2\alpha}{\mathrm{u}}]cosx$$

This is the second-order ordinary differential equation have the particular solution in the form:

$$V(x,s,u)=\frac{-[\frac{1}{u^2\mathrm{s}}+\frac{1}{\mathrm{u}}]cosx}{\alpha^2\mathrm{D}^2-\left[\frac{1}{u^2s^2}+2\alpha\frac{1}{\mathrm{us}}\right]}=\frac{-[\frac{1}{u^2\mathrm{s}}+\frac{2\alpha}{\mathrm{u}}]cosx}{-\alpha^2-\left[\frac{1}{u^2s^2}+2\alpha\frac{1}{\mathrm{us}}\right]}=\frac{-[\mathrm{s}+2\alpha\mathrm{us}^2]cosx}{-u^2s^2\alpha^2-[1+2\alpha\mathrm{us}]}$$

$$= \frac{[s+2\alpha u s^2] cosx}{[1+\alpha us]^2} = [\frac{s}{[1+\alpha us]} + \frac{\alpha u s^2}{[1+\alpha us]^2}] \cos x \quad (26)$$

Taking inverse Abaoub- Shkheam transform in(26) we get

$$u(x,t)=(1+\alpha t)e^{-\alpha t} \cos x.$$

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