حل موجات الظل المعمم لنظام القيم الابتدائية من نوع ريمان

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الملخص

في هذه الورقة، ندرس نظامًا غير مفرط في قوانين الحفظ عند وجود اللزوجة. نحن لا نستخدم الحل الكلاسيكي. نقوم ببناء حل موجة الظل (انظر [9]) للبيانات الأولية من نوع ريمان حيث المكون الأول يطور الخلخلة والتي تم تحديدها جزئيًا في [5] باستخدام طريقة لزوجة التلاشي. بالنسبة لنهج موجة الظل، استخدمنا فقط الانتروبيا المحدبة وتدفق الانتروبيا المقابل لها.

Generalized shadow waves solution to a system of Riemann type initial data

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Abstract

In this paper, we study a system of a non-strictly hyperbolic of conservation laws when viscosity is present. We do not use classical solution. We construct a shadow wave solution (see [9]) for Riemann type initial data where the first component develops rarefaction, which was partially determined in [5] using vanishing viscosity method. For shadow wave approach, we used only convex entropy and its corresponding entropy.

1-Introduction:

Conservation laws in applications are sometimes not strictly hyperbolic. A non-strictly hyperbolic system of conservation laws when viscosity is present and when viscosity is zero, which has been studied in [5]. It was partially determined as vanishing viscosity limit in [5].

The systems of nonlinear partial differential equations of the form

$$(u_j)_t + \sum_{i=1}^{j}\left(\frac{u_i\ u_{j-i+1}}{2}\right)_x = \frac{\varepsilon}{2}(u_j)_{xx}, \quad j = 1,2,\ldots..n \qquad (1)$$

where $\varepsilon > 0$ is a small parameters with initial condition of the following form

$$u_j(x,0) = u_{j0}(x), \quad j = 1,2,\ldots..n \qquad (2)$$

This system (1) is introduced in [3]. It is shown there that the system (1) can be linearized by using a generalized Hopf–Cole transformation, this in turn gives explicit formula for $u_i, i = 1,2, \ldots n$.

It is well-known that the corresponding inviscid partial differential equations system

$$(u_j)_t + \sum_{i=1}^{j} \left(\frac{u_i\, u_{j-i+1}}{2}\right)_x = 0, \quad j = 1,2, \ldots\ldots n \qquad (3)$$

is not strictly hyperbolic as it has repeated eigenvalues, i.e., $\lambda_i = (u_1, u_2, \ldots., u_n) = u_1$, for $i = 1,2, \ldots\ldots n$.. The above inviscid system does not have smooth global solution, even if the initial data (2) is smooth. Additional conditions are required to pick the unique physical solution. Vanishing viscosity method is one of ways to select the physical weak solution of (3). That is, the solution of the inviscid system is constructed as the limit goes to zero of solutions $u_j(x,t)$ of (1), with suitable initial and boundary conditions.

For n = 4 of the system (1), with Riemann type initial data for the case where the first component develops a shock. With $u = u_1$, $v = u_2$, $w = u_3$ and $z = u_4$, with coefficient of viscosity being a generalized constant γ, then the system (1), becomes

$$u_t + \left(\frac{u^2}{2}\right)_x = \frac{\gamma}{2} u_{xx}, \qquad v_t + (uv)_x = \frac{\gamma}{2} v_{xx}$$

$$w_t + \left(\frac{v^2}{2} + uw\right)_x = \frac{\gamma}{2} w_{xx}, \qquad z_t + (vw + uz)_x = \frac{\gamma}{2} z_{xx} \qquad (4)$$

with initial data

$$(u(x,0), v(x,0), w(x,0), z(x,0)) = (u_0, v_0, w_0, z_0).$$

In [5], the vanishing viscosity limit for Riemann type initial data is studied for the case n = 4. The main purpose of this paper is to solve the Riemann type initial data when u develops rarefaction($u_0 < u_1$) for the components w and z of the system (4). We construct a shadow wave solution (see [9]) for Riemann type initial data where the first component develops rarefaction, which was partially determined in [5] using vanishing viscosity method. The shadow waves are defined by nets of piecewise constant functions for time variable t fixed parameterized by some small parameter ε > 0 and bounded in L^1_{loc}(R).

2-General formulas and entropy conditions:

Definition 1. Let U_ε and $U|_{t=0}$ be given by

$$U_\varepsilon\,(x,t) = \begin{cases} U_0\,, & x < c(t) - a_\varepsilon(t) - x_{1,\varepsilon} \\ U_{1,\varepsilon}\,, & c(t) - a_\varepsilon(t) - x_{1,\varepsilon} < x < c(t) \\ U_{2,\varepsilon}\,, & c(t) < x < c(t) - b_\varepsilon(t) - x_{2,\varepsilon} \\ U_1\,, & x > c(t) - b_\varepsilon(t) - x_{2,\varepsilon} \end{cases} \quad (5),$$

$$U|_{t=0} = \begin{cases} U_0, & x < 0 \\ U_1, & x > 0 \end{cases} \quad (6)$$

Here $x_{1,\varepsilon}, x_{2,\varepsilon} \sim \varepsilon$, where $U_0, U_{1,\varepsilon}, U_{2,\varepsilon}$ and U_1 are constants that are in R^n, $(x, t) \in R \times (0, \infty)$ and f : $R^n \to R$ is smooth function. The line $x = c(t)$ has its initial point at origin. Assume that the distributional limit of $U_\varepsilon\,(x,t)$ exists and equals to U. If $(U_\varepsilon)_t + f(U_\varepsilon)_x$ tends to 0, in the sense of distribution, then we say U is a Shadow wave solution to the conservation law

$$(U)_t + f(U_\varepsilon)_x = 0 \quad (7)$$

with initial data

$$U(x,0) = U|_{t=0}.$$

Definition 2. Let U_ε be given by

$$U_\varepsilon\,(x,t) = \begin{cases} U_0\,, & x < (c - a_\varepsilon)t \\ U_{1,\varepsilon}\,, & (c - a_\varepsilon)t < x < ct \\ U_{2,\varepsilon}\,, & ct < x < (c + b_\varepsilon)t \\ U_1\,, & x > (c + b_\varepsilon)t \end{cases} \quad (8)$$

Then

$$\begin{aligned} \partial_t f(U_\varepsilon) &\approx -c\,(f(U_1) - f(U_0)\delta - c\big(a_\varepsilon f\big(U_{1,\varepsilon}\big) + b_\varepsilon f\big(U_{2,\varepsilon}\big)\big)t\delta' \\ &\quad + \big(a_\varepsilon f\big(U_{1,\varepsilon}\big) + b_\varepsilon f\big(U_{2,\varepsilon}\big)\big)\delta \\ \partial_x g(U_\varepsilon) &\approx (g(U_1) - g(U_0)\delta - \big(a_\varepsilon g\big(U_{1,\varepsilon}\big) + b_\varepsilon g\big(U_{2,\varepsilon}\big)\big)t\delta' \end{aligned} \quad (9)$$

is a special case of (5) which is general enough for solving Riemann problem (2), We will call it the *simple shadow wave.*

Definition 3. Let $\eta(U)$ be a convex entropy with the entropy flux $q(U)$ for the system (7). Then U_ε is said to be entropy admissible if

$$\lim_{\varepsilon\to 0} inf \int_R \int_0^T \eta(U_\varepsilon)\partial_t \emptyset dxdt + \mathrm{q}(U_\varepsilon)\partial_x \emptyset + \int_R \eta(U_\varepsilon(x,t))\,dx \geq 0$$

for all non-negative test functions $\emptyset \in C_0^\infty$ *(R*$\times$ $(0, T)$*).*

The above definition is equivalent to:

$$\limsup_{\varepsilon\to 0} (-c\left(\eta(U_1)-\eta(U_0)\right)+\varepsilon\left(\eta\left(U_{2,\varepsilon}\right)+\eta\left(U_{1,\varepsilon}\right)\right)+q(U_1)-q(U_0)\leq 0,$$

and

$$\lim_{\varepsilon\to 0}(-c\,\varepsilon\left(\eta\left(U_{2,\varepsilon}\right)-\eta\left(U_{1,\varepsilon}\right)\right)+\varepsilon\left(q\left(U_{2,\varepsilon}\right)-q\left(U_{1,\varepsilon}\right)\right)=0,$$

where here $a_\varepsilon, b_\varepsilon \sim \varepsilon$.

Vanishing viscosity limit -the rarefaction case.

By vanishing viscosity limit, see [5] the limit *(u, v)* for the rarefaction case of *u* is given by

$$(u,v)=\begin{cases}(u_0,v_0), & x<u_0t\\ \left(\frac{x}{t},0\right), & u_0t< x<u_0t\\ (u_1,v_1), & x>u_0t\end{cases} \qquad (10)$$

With the above definitions, we have the following theorem.

3-Shadow wave solution for Riemann type initial data:

Theorem 1. *The system (4) has shadow wave solution when u develops rarefaction* $u_0 < u_1$, *for the components w and z is given by*

$$w_t = w_0 + \frac{v_0^2}{2}t\delta_{x=u_0\,t} - \frac{v_1^2}{2}t\delta_{x=u_1\,t} + w_1$$

$$z_t = z_0 + u_0w_0t\delta_{x=u_0\,t} - u_1w_1t\delta_{x=u_1\,t} + z_1$$

with initial data

$$(u(x,0),\ v(x,0),\ w(x,0),\ z(x,0)\ ) = \begin{cases}u_0,v_0,w_0,z_0, & x>0\\ u_1,v_1,w_1,z_1, & x<0\end{cases}$$

Proof: We suppose the following converges for *(u, v, w, z)* for the possible shadow wave approximation in which solutions taken for the components *u* and *v* is vanishing viscosity limit.

$$(u_\varepsilon,v_\varepsilon,w_\varepsilon,z_\varepsilon)(x,t)=\begin{cases}(u_0,v_0,w_0,z_0)\,, & x<(u_0-\varepsilon)t\\ (u_{1,\varepsilon},\frac{v_{1,\varepsilon}}{\sqrt{\varepsilon}},\frac{w_{v_{1,\varepsilon}}}{\sqrt{\varepsilon}},\frac{z_{v_{1,\varepsilon}}}{\sqrt{\varepsilon}})\,, & (u_0-\varepsilon)t< x<u_0t\\ \left(\frac{x}{t},0\,,0,0\right), & u_0t< x<u_1t\\ (u_{2,\varepsilon},\frac{v_{2,\varepsilon}}{\sqrt{\varepsilon}},\frac{w_{v_{2,\varepsilon}}}{\sqrt{\varepsilon}},\frac{z_{v_{2,\varepsilon}}}{\sqrt{\varepsilon}}), & u_1t< x< (u_1+\varepsilon)t\\ (u_1,v_1,w_1,z_1\,), & x<(u_1+\varepsilon)t\end{cases}$$

We will use the formula of the simple shadow wave (9), under the assumption,

$$|f(U_{i,\varepsilon})|_{L^\infty},\ |g(U_{i,\varepsilon})|_{L^\infty} = O\left(\frac{1}{\varepsilon}\right), \qquad i = 1,2,$$

then, we have the following formula for distributional derivatives.

$$\partial_t f(U_\varepsilon) \approx -c\,(f(U_1) - f(U_0)\delta - c\big(a_\varepsilon f\big(U_{1,\varepsilon}\big) + b_\varepsilon f\big(U_{2,\varepsilon}\big)\big)t\delta' + \big(a_\varepsilon f\big(U_{1,\varepsilon}\big) + b_\varepsilon f\big(U_{2,\varepsilon}\big)\big)\delta$$

$$\partial_x g(U_\varepsilon) \approx (g(U_1) - g(U_0)\delta - \big(a_\varepsilon g\big(U_{1,\varepsilon}\big) + b_\varepsilon g\big(U_{2,\varepsilon}\big)\big)t\delta' \qquad (12)$$

Applying the above formula (12) with assume that $a_\varepsilon = \varepsilon,\ b_\varepsilon = 0,\ c = u_0$, near the discontinuity line $x= u_0\, t$ and $a_\varepsilon = 0,\ b_\varepsilon = \varepsilon,\ c = u_0$, near the discontinuity line $x= u_1\, t$, then we get

$$w_t \approx \big(u_0 w_0 + w_{1,\varepsilon}\big)\delta_{x=u_0 t} - u_0 w_{1,\varepsilon} t\delta'_{x=u_0 t} + \big(-u_1 w_1 + w_{2,\varepsilon}\big)\delta_{x=u_1 t} - u_1 w_{2,\varepsilon} t\delta'_{x=u_1 t}$$

$$\partial_x\left(\frac{v_\varepsilon^2}{2} + u_\varepsilon w_\varepsilon\right) \approx \left(-\frac{v_0^2}{2} + u_0 w_0\right)\delta_{x=u_0 t} + \left(-\frac{v_{1,\varepsilon}^2}{2} + u_0 w_{1,\varepsilon}\right)t\delta'_{x=u_0 t} + \left(-\frac{v_1^2}{2} + u_1 w_1\right)\delta_{x=u_1 t} + \left(-\frac{v_{2,\varepsilon}^2}{2} + u_1 w_{2,\varepsilon}\right)t\delta'_{x=u_1 t}$$

The sign ,, ≈,, simple means a convergence to zero as $\varepsilon \to 0$.

Since the relation $w_t + \partial_x\left(\frac{v_\varepsilon^2}{2} + u_\varepsilon w_\varepsilon\right) \approx 0$, then that implies

$$v_{1,\varepsilon} = v_{2,\varepsilon} = 0,\ \ w_{1,\varepsilon} = \frac{v_0^2}{2},\ \ w_{2,\varepsilon} = -\frac{v_1^2}{2}$$

Now we calculate the distributional limit of w_ε. Let ∅ be a real valued test function supported in $R \times (0, \infty)$, multiplying with a test function ∅ in the equation for component w in (4) and integrating over $R \times (0, T)$, we have

$$\int_{0}^{\infty}\int_{-\infty}^{\infty} w_{\varepsilon}(x,t)\emptyset(x,t)\,dx\,dt$$
$$= \int_{0}^{\infty}\int_{-\infty}^{(u_0-\varepsilon)t} w_{\varepsilon}(x,t)\emptyset(x,t)\,dx\,dt$$
$$+ \int_{0}^{\infty}\int_{(u_0-\varepsilon)t}^{u_0 t} w_{\varepsilon}(x,t)\emptyset(x,t)\,dx\,dt$$
$$+ \int_{0}^{\infty}\int_{u_1 t}^{(u_1+\varepsilon)t} w_{\varepsilon}(x,t)\emptyset(x,t)\,dx\,dt$$
$$+ \int_{0}^{\infty}\int_{(u_1+\varepsilon)t}^{\infty} w_{\varepsilon}(x,t)\emptyset(x,t)\,dx\,dt \qquad (13)$$

For the left hand side, we have

$$\int_{0}^{\infty}\int_{-\infty}^{\infty} w_{\varepsilon}(x,t)\emptyset(x,t)\,dx\,dt = \frac{\gamma}{2}\int_{0}^{\infty}\int_{-\infty}^{\infty} w_{xx}(x,t)\emptyset(x,t)\,dx\,dt \qquad (14)$$

Using integration by parts twice on the right hand side of the equation (5), we have

$$\int_{0}^{\infty}\int_{-\infty}^{\infty} w_{\varepsilon}(x,t)\emptyset(x,t)\,dx\,dt = \frac{\gamma}{2}\int_{0}^{\infty}\int_{-\infty}^{\infty} w(x,t)\emptyset(x,t)_{xx}\,dx\,dt \quad .(15)$$

By the assumption

$$\sup_{x\in R, 0<x<\infty} |\, w(x,t)| = o\left(\frac{\gamma}{2}\right)^{-1} \qquad (16)$$

The right hand side of equation (15) implies

$$\lim_{\varepsilon\to 0}\int_{0}^{\infty}\int_{-\infty}^{\infty} w_{\varepsilon}(x,t)\emptyset(x,t)\,dx\,dt = \int_{0}^{\infty}\int_{-\infty}^{\infty} w(x,t)\emptyset(x,t)\,dx\,dt$$

So, that

$$\lim_{\varepsilon\to 0}\int_{0}^{\infty}\int_{-\infty}^{\infty} w_{\varepsilon}(x,t)\emptyset(x,t)\,dx\,dt = 0$$

By (14), the above limit tends to zero as $\frac{\gamma}{2}$ tends to zero.

After above calculations, the equation (13) implies that

$$\int_0^{\infty}\int_{-\infty}^{\infty} w_\varepsilon(x,t)\emptyset(x,t)\,dx\,dt$$
$$= \int_0^{\infty}\int_{-\infty}^{(u_0-\varepsilon)t} w_0\emptyset(x,t)\,dx\,dt + \int_0^{\infty}\int_{(u_0-\varepsilon)t}^{u_0 t} \frac{v_0^2}{2\varepsilon}\emptyset(x,t)\,dx\,dt + \int_0^{\infty}\int_{u_1 t}^{(u_1+\varepsilon)t} \frac{v_1^2}{2\varepsilon}\emptyset(x,t)\,dx\,dt + \int_0^{\infty}\int_{(u_1+\varepsilon)t}^{\infty} w_1\,\emptyset(x,t)\,dx\,dt = 0$$

As ε tends to 0, we get the formula for w in (11).
The similar analysis as in w for the component z, then we get

$$z_t \approx \left(u_0 z_0 + z_{1,\varepsilon}\right)\delta_{x=u_0\,t} - u_0 z_{1,\varepsilon} t\delta'_{x=u_0\,t} + \left(-u_1 z_1 + z_{2,\varepsilon}\right)\delta_{x=u_1\,t} - u_1 z_{2,\varepsilon} t\delta'_{x=u_1\,t}$$

$$\partial_x(v_\varepsilon w_\varepsilon + u_\varepsilon z_\varepsilon) \approx (-v_0 w_0 + u_0 z_0)\delta_{x=u_0\,t} + \varepsilon\left(-\frac{v_{1,\varepsilon} w_{1,\varepsilon}}{\varepsilon^{\frac{3}{2}}} + \frac{u_0 z_{1,\varepsilon}}{\varepsilon}\right) t\delta'_{x=u_0\,t}$$
$$+(-v_1 w_1 + u_1 z_1)\delta_{x=u_1\,t} + \varepsilon\left(-\frac{v_{2,\varepsilon} w_{2,\varepsilon}}{\varepsilon^{\frac{3}{2}}} + \frac{u_1 z_{2,\varepsilon}}{\varepsilon}\right) t\delta'_{x=u_1\,t}$$

Since from the above calculation for w, we had $v_{1,\varepsilon}, v_{2,\varepsilon} = 0$. So

$$\partial_x(v_\varepsilon w_\varepsilon + u_\varepsilon z_\varepsilon) \approx (-v_0 w_0 + u_0 z_0)\delta_{x=u_0\,t} + u_0 z_{1,\varepsilon} t\delta'_{x=u_0\,t} + (v_1 w_1 - u_1 z_1)\delta_{x=u_1\,t} + u_1 z_{2,\varepsilon} t\delta'_{x=u_1\,t}$$

The relation $z_t + \partial_x(v_\varepsilon w_\varepsilon + u_\varepsilon z_\varepsilon) \approx 0$, then that implies

$$z_{1,\varepsilon} = v_0 w_0,\ \ z_{2,\varepsilon} = -v_1 w_1.$$

With the same calculation as in w_ε for z_ε, we get

$$\lim_{\varepsilon\to 0} \int_0^\infty \int_{-\infty}^\infty z_\varepsilon(x,t)\emptyset(x,t)\,dx\,dt = \int_0^\infty \int_{-\infty}^\infty z(x,t)\emptyset(x,t)\,dx\,dt \qquad (17)$$

Then we have

$$\begin{aligned}\int_0^\infty \int_{-\infty}^\infty z_\varepsilon(x,t)\emptyset(x,t)\,dx\,dt &= \int_0^\infty \int_{-\infty}^{(u_0-\varepsilon)t} z_0\emptyset(x,t)\,dx\,dt \\ &+ \int_0^\infty \int_{(u_0-\varepsilon)t}^{u_0 t} v_0 w_0\,\emptyset(x,t)\,dx\,dt \\ &+ \int_0^\infty \int_{u_1 t}^{(u_1+\varepsilon)t} -v_1 w_1\,\emptyset(x,t)\,dx\,dt \\ &+ \int_0^\infty \int_{(u_1+\varepsilon)t}^{\infty} z_1\,\emptyset(x,t)\,dx\,dt = 0\end{aligned}$$

As ε tends to 0, we get the formula for z in (11).

For general type Riemann initial data, we used shadow wave approach to construct solution. We note that shadow wave solution for the component w agrees with the vanishing viscosity limit but the component z does not agree for the above special type Riemann initial data. For shadow wave approach we used only convex entropy and its corresponding entropy flux.

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