توزيع سجل ليندلي الجديد مع التطبيقات

فاطمة سعيد علي الزوق - كلية الآداب الجميل جامعة صبراتة

ملخص البحث :

الخلاصة: تقدم هذه الورقة تعميماً جديداً عن توزيع ليندلي الذي قدمه ليندلي (1958)، باستخدام الفكرة الأساسية لباباس وآخرين (2012) وعلى غرار مارشال وأولكين (1997) التوزيع الجديد هو مركب من توزيعي Lindley و Logarithmic . نشير إلى النموذج الجديد باسم التوزيع اللوغاريتمي - ليندلي (Log-L). هذا النموذج قادر على نمذجة أشكال مختلفة من معايير الشيخوخة والفشل. تمت مناقشة خصائص النموذج الجديد ويتم استخدام أقصى احتمالية وأقصى مسافات بين المنتج وأساليب تقدير المربعات الصغرى لتقييم المعلمات المعنية. يتم اشتقاق التعبيرات الصريحة للحظات وفحص إحصائيات النظام. أخيرًا، تم توضيح فائدة النموذج الجديد لنمذجة بيانات الموثوقية باستخدام مجموعتي بيانات حقيقية مع دراسة محاكاة.

A New Log Lindley Distribution with Applications

Fatimah Saeid Ali Alzawq

Abstract

This paper introduces a new generalization of the Lindley distribution introduced by Lindley (1958), using the basic idea of Pappas et al. (2012), and along the lines of Marshall and Olkin (1997). The new distribution is a compound of the Lindley and Logarithmic distributions. We refer to the new model as the Logarithmic-Lindley (Log-L) distribution. This model is capable of modeling various shapes of aging and failure criteria. The properties of the new model are discussed and the maximum likelihood, maximum product spacings and least square estimation methods are used to evaluate the parameters involved. Explicit expressions are derived for the moments and examine the order statistics. Finally, the usefulness of the new model for modeling reliability data is illustrated using a two real data sets with simulation study.

Keywords: Lindley distribution; Logarithmic distribution; maximum

likelihood estimation; maximum product spacing's estimates; least square estimators.

1 Introduction

Lifetime distribution represents an attempt to describe, mathematically, the length of the life of systems or devices. Lifetime distributions are most frequently used in many fields as medicine, engineering ...etc. Many parametric models such as exponential, gamma and Weibull have been frequently used in statistical literature to analyze lifetime data. But there is no clear motivation for the gamma and Weibull distributions. They only have more general mathematical closed form than the exponential distribution with one additional parameter.

Recently, the one parameter Lindley distribution has attracted the researchers for its use in modeling lifetime data. It has been observed in several papers that this distribution has performed excellently. The Lindley distribution was originally proposed by Lindley in 1958 in the context of Bayesian statistics, as a counter example of fiducial statistics. One can glean it as a mixture of exponential(θ) and gamma (2, θ).

Some of the advances in the literature of Lindley distribution are given by Ghitany et al. (2011) who has introduced a two-parameter weighted Lindley distribution and has pointed that Lindley distribution is particularly useful in modeling biological data from mortality studies. Mahmoudi et al. (2010) have proposed generalized Poisson Lindley distribution. Bak- ouch et al. (2012) have come up with extended Lindley (EL) distribution, Adamidis and Loukas (1998) have introduced exponential geometric (EG) distribution. Shanker et al. (2013) have introduced a two-parameter Lindley distribution. Zakerzadeh et al. (2012) have proposed a new two parameter lifetime distribution: model and properties. M.K. Hassan (2008) has introduced a convolution of Lindley distribution. Ghitany et al. (2013) worked on the estimation of the reliability of a stress-strength system from power Lindley distribution. Elbatal et al. (2013) has also proposed a new generalized Lindley distribution.

Definition 1, A random variable X is said to have the Lindley distribution with parameter (θ) if its probability density function (pdf) is defined as

$$\text{g (x)} = \frac{\theta^2}{\theta+1}(1+\ \chi)e^{-\theta\chi}, \chi > 0, \theta > 0 \qquad (1)$$

while the corresponding survival, or reliability, function is given by

$$\overline{G}\,(x) = \frac{\theta + 1 + \theta^{x}}{\theta} e^{-\theta x}, x \quad > 0. \qquad (2)$$

also, the hazard rate function, denoted by $r(x)$, is given by

$$r(x) = \frac{\theta^2(1 + x)}{\theta + 1 + \theta^{x}}\ , x > 0. \qquad (3)$$

More details about the Lindley distribution can be found in Ghitany et al. (2008).

In the context of reliability and survival analysis, Marshall and Olkin (1997) proposed a transformation of a distribution $G\,(x; \theta)$ that introduces a new parameter α > 0. This transformation is defined through the cumulative distribution function (cdf).

$$f(x; \theta, \alpha) \quad = \frac{G(x; \theta)}{G(x; \theta) + \alpha\overline{G}(x; \theta)}. \qquad (4)$$

The interpretation of the parameter α is given in Marshall and Olkin (1997) in terms of the behavior of the ratio of hazard rates of G and F. This ratio is increasing in x for $\alpha \geq 1$ and decreasing in x for $0 < \alpha < 1$. This transformation is then proposed for the Exponential and Weibull distribution in Marshall and Olkin (1997) in order to generate more flexible models for lifetime data. Clearly, for α = 1, F and G coincide.

A lot of papers had been published by using Marshall-Olkin (M-O) transformation given in (4). Alice and Jose (2003) introduced M-O extended semi-Pareto model and studied its geometric extreme stability. Semi-Weibull distribution and generalized Weibull distributions are considered by Alice and Jose (2005). M-O extended Pareto distribution was introduced by Ghitany (2005). Ristic et al. (2007) introduced and studied the M-O gamma distribution. Ghitany et al. (2007) proposed the M-O extended Lomax distribution to apply with censored data. The M-O beta distribution as an extension of the basic distribution with four parameters was presented by Jose et al. (2009). Gomez-Deniz (2010) presented a new generalization of the geometric distribution using the M-O scheme. Garcia et al. (2010) define a generalized normal distribution by applying this transformation to a normal distribution G.

In 2012, Pappas et al. introduced a new generalization which is derived along the lines of Marshall and Olkin (1997). Accordingly, starting with a survival function$\overline{G}(x)$, then the usual device of adding a new parameter results in another survival function $\overline{F}(x)$ defined by

$$\overline{F}(x) = \frac{\ln[\,1-(1-p)\overline{G}(x)]}{\ln(p)};x\in\mathbb{R},p>0 \qquad (5)$$

and when $p\to 1$ the distribution reduces to the base distribution $\overline{G}$(x). If $f(x)$ and $h(x)$ are the pdf and hazard rate function corresponding to $\overline{F}(x)$, then

$$f(x) = \frac{(p-1)\,g(x)}{[\,1-(1-\mathrm{p})\overline{\mathrm{G}}(\mathrm{x})]\ln(p)};x\in\ \mathbb{R},p>0 \qquad (6)$$

and

$$h(x) = \frac{(p-1)\overline{G}(x)r(x)}{[\,1-(1-\mathrm{p})\overline{\mathrm{G}}(\mathrm{x})]\,\ln[1-(1-p)\overline{G}(x)]}, \qquad (7)$$

where h(x) is the hazard rate corresponding to f(x). It is worth mentioning that Al-Zahrani and Sagor (2014) followed this idea to provide the Lomax-Logarithmic distribution. The aim of this paper is to introduce a new generalization of Lindley (1958) distribution. This generalization is flexible enough to model different types of lifetime data having different forms of failure rate. The new model can accommodate both decreasing and increasing failure rates as its antecessors, as well as unimodal and bathtub shaped failure rates.

The rest of this paper will cover the following topics adequately: Section 2 introduces the pdf and the survival function of the Logarithmic-Lindley distribution, then gives an interpretation of the new model. We investigate the reliability analysis of the new model via Section 3 which includes the hazard rate function with its shapes, the cumulative hazard rate function and the mean residual lifetime. Section 4 presents the statistical properties including the moments, moment generating function and quantile function. Section 5 demonstrates the distribution of order statistics. Section 6 introduces the Lorenz and Bonferroni curves as measures of inequality besides the Renyi entropy as an important measure of uncertainty. Section 7 states the different methods of parameter estimation such as the maximum likelihood estimation method, maximum product spacing estimates and least square estimates. Section 8 provides two applications illustrating the performance of the new proposed model that are applied on different real data sets. Finally, Section 9 presents some conclusions.

2 A Lindley Extension Model

In the following, Lindley distribution is extended by adding a new shape parameter, p > 0, using Equations (5) through (7). Now, substituting (2) into (5) and doing the necessary simplifications gives the survival function of the Logarithmic-Lindley (Log-L) distribution as

$$\overline{F}(x) = \frac{\ln\left[1-(1-\mathrm{p})(\frac{1+\theta+\theta x}{\theta+1})e^{-\theta x}\right]}{\ln(p)}; \ \ x>0, \tag{8}$$

where θ> 0 is a scale parameter and p > 0 is a shape parameter. Then the pdf corresponding to (8) is readily found to be

$$f(x) = \frac{\theta^2(p-1)}{(\theta+1)\ln(p)}\left[\frac{(1+x)e^{-\theta x}}{1-(1-p)\left(\frac{1+\theta+\theta x}{\theta+1}\right)e^{-\theta x}}\right]; x>0,\theta,p > 0. \tag{9}$$

Note that the Logarithmic-Lindley distribution is an extended model to analyze more complex data and it generalizes some of the widely used distributions. The Lindley distribution is clearly a special case when p → 1. INTERPRETATION For p ∈(0,1), the pdf given by (9) can be obtained as a compound of the Logarithmic and the Lindley distributions. According to Barlow and Proschan (1996) and Arnold et al. (1992), suppose that X_1,X_2, ...,X_y are Y iid (independent and identically distributed) lifetime random variables in a series system each with pdf (1), and let Y be a random variable distributed according to the Logarithimc distribution with probability mass function (pmf) defined as

$$p(Y=y) = \frac{-(1-p)^y}{ylnp}; y \in \mathbb{N}, p \in (0,1).$$

Now, the conditional distribution function of (X | Y) is given by

$$f(x \mid y) = yg(x)\left[\overline{G}(x)\right]^{y-1} = \frac{y\theta^2(1+x)}{1+\theta+\theta^2}\ (\frac{1+\theta+\theta x}{1+\theta}\ e^{-\theta x})^y\ ,$$

where $g(x)$ and $\overline{G}(x)$ are the pdf and the survival function corresponding to Lindley distribution and given by (1) and (2), respectively.

Then, the joint distribution of the random variables X and Y, denoted by f (x, y), is obtained as

$$f(x,y) = (x \mid y).p(Y = p)$$
$$= \frac{-\theta^2(1+x)}{(1+\theta+\theta x)\ln(p)}\left[(1-p)\left(\frac{1+\theta+\theta x}{1+\theta}\right)e^{-\theta x}\right]^y$$

Hence, it can be found the marginal pdf of x as follows

$$f(x) = \sum_{y=1}^{\infty} f(x,y)$$
$$= \frac{-\theta^2(1+x)}{(1+\theta+\theta x)\ln(p)}\sum_{y=1}^{\theta}\left[(1-p)\left(\frac{1+\theta+\theta x}{1+\theta}\right)e^{-\theta x}\right]^y$$
$$= \frac{-\theta^2(1+x)}{(1+\theta+\theta x)\ln(p)}\left[\frac{(1-p)\left(\frac{1+\theta+\theta x}{1+\theta}\right)e^{-\theta x}}{1-(1-p)\left(\frac{1+\theta+\theta x}{1+\theta}\right)e^{-\theta x}}\right]^y$$
$$= \frac{\theta^2(p-1)}{(1+\theta)\ln(p)}\left[\frac{(1+x)e^{-}\theta x}{1-(1-p)\left(\frac{1+\theta+\theta x}{\theta+1}\right)e^{-\theta x}}\right],$$

which is the pdf of the Log-L distribution given by (1).

Figure 1 illustrates some of the possible shapes of the pdf of the logarithmic-Lindley distribution for different values of the parameters θ and p chosen from the ranges specified in equation (9).

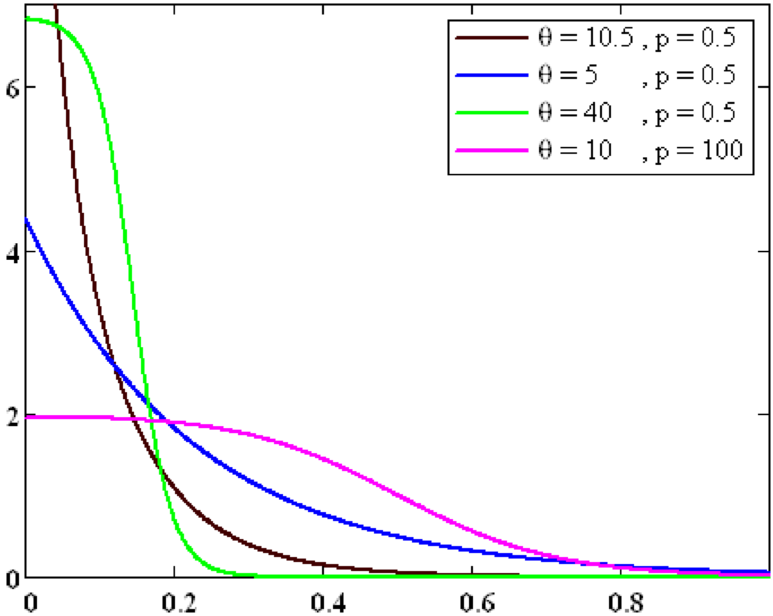


FIGURE 1: The density function of the Logarithmic-Lindley distribution.

3 Reliability Analysis

In this section, we present the hazard rate function with its different shapes, the cumulative hazard rate function and the mean residual lifetime for the Logarithmic-Lindley distribution.

3.1 The Hazard Rate Function

Let X be the lifetime of a device (or a component in a system). Suppose a component follow that X has a pdf as in (9). One of the most important characteristics of X is its hazard rate function $h(\mathrm{x})$ defined by

$$h(x) = \lim_{\Delta x \to 0} \frac{Pr(x < X < x + \Delta x|\mathrm{X} > \mathrm{x})}{\Delta x} = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x . R(x)}$$
$$= \frac{f(x)}{R(x)},$$

which provides information about a small interval after time $\mathrm{x}(\mathrm{x} + \Delta\mathrm{x})$. Using the previous definition or by substituting (2) and (3) into (7), the hazard rate function of a random variable $\mathrm{X}\tilde{}\mathrm{Log} - \mathrm{L}(\theta, \mathrm{p})$is given by

$$h(x) = \frac{(p-1)\left(\frac{\theta^2}{\theta+1}\right)(1+x)e^{-\theta x}}{\left[1-(1-p)\left(\frac{\theta+1+\theta x}{\theta+1}\right)e^{-\theta x}\right]\ln[1-(1-p)(\frac{\theta+1+\theta x}{\theta+1})e^{-\theta x}]}$$

By taking the limit of (10) when x → 0and when x → ∞ as follows

$$\lim_{x\to 0} h(x) = \frac{p-1}{p\ln(p)} \times \frac{\theta^2}{\theta+1} = \frac{p-1}{p\ln(p)} \lim_{x\to 0} r(x),$$

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} r(x)$$

it follows from (10) that

$$\frac{p-1}{p\ln(p)} r(x) \le h(x) \le r(x); x > 0, p \ge 1,$$

and

$$r(x) \le h(x) \le \frac{p-1}{p\ln(p)} r(x); x > 0, p \in (0,1).$$

Hence, using the ratio $\frac{\mathrm{h(x)}}{\mathrm{r(x)}}$, $x > 0$, it can be shown that $\frac{h(x)}{r(x)}$ is increasing for $p \ge 1$ and decreasing for $p \in (0,1)$. Figure 2 illustrates the behavior of the hazard rate function of the Logarithmic-Lindley distribution at different values of the parameters involved.

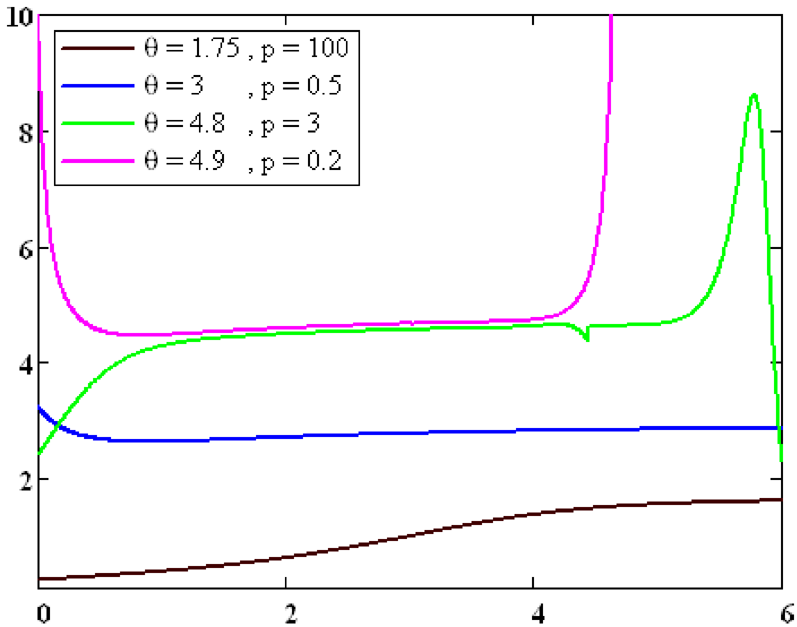


FIGURE 2: Increasing, decreasing, upside down and bathtub shapes of the hazard rate function for the Log-L distribution.

Therefore, the new distribution can accommodate both decreasing and increasing failure rates as its antecessors, as well as unimodal and bathtub shaped failure rates.

3.2 The Cumulative Hazard Rate Function

Many generalized models have been proposed in reliability literature through the relationship between the reliability function $\overline{F}(x)$ and its cumulative hazard rate function H(x) given by $H(x) = -\ln \overline{F}\ (x)$. Then, the cumulative hazard rate function of the Logarithmic-Lindley distribution is given by

$$H(x) = \ln[\ln(p)] - \ln\left[\ln\left\{1 - (1-p)\left(\frac{\theta + 1 + \theta x}{\theta + 1}\right)e^{-\theta x}\right\}\right] \qquad (11)$$

3.3 The Mean Residual Lifetime

The additional lifetime given that the component has survived up to time x is called the residual life function of the component, then the expectation of the random variable X_x that represent the remaining lifetime is called the mean residual lifetime (MRL) and is given by

$$m(x) = E(X - x|X \geq x) = \{\frac{1}{\overline{F}(x)}\int_x^{\infty} t.f(t)dt\} - x\} \quad (12)$$

While the hazard rate function $h(x)$ provides information about a small interval after time x $(just\ after\ x)$, the MRL considers information about the whole interval after x $(all\ after\ x)$. The MRL as well as the hazard rate function or the reliability function is very important as each of them can be used to characterize a unique corresponding lifetime distribution.

The MRL function $m(x)$ for Logarithmic-Lindley random variable can be derived in the following steps.

Now,

$$\int_x^{\infty} t.f(t)dt$$

$$= \frac{\theta^2(p-1)}{(\theta+1)\ln(p)}\int_x^{\infty}\frac{(t+t^2)e^{-\theta t}}{\left[1-(1-p)\left(1+\frac{\theta t}{\theta+1}\right)e^{-\theta t}\right]}dt \quad (13)$$

Using the expansion $(1-z)^{-1} = \sum_{j=0}^{\infty} zi$, $|z| < 1$, $one\ has$

$$\left[1-(1-p)(1+\frac{\theta t}{\theta+1})e^{-\theta t}\right]^{-1}$$

$$= \sum_{j=0}^{\infty}(1-p)^j\,(1+\frac{\theta t}{\theta+1})^{\,j}e^{-j\theta t} \quad (14)$$

Similarly, using the expansion $(1+b)^n = \sum_{l=0}^{\infty}\binom{j}{i}$, one can have

$$(1+\frac{\theta+t}{\theta+1}^{j}) = \sum_{i=0}^{\infty}\binom{j}{i}(\frac{\theta}{\theta+1})^{j-i}\ t^{j-i} \quad (15)$$

Hence, one can rewrite (14) as

$$[1-(1-p)(1+\frac{\theta t}{\theta+1})e^{-\theta t}]^{-1} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{j}{i}(1-p)^{j}\,(\frac{\theta}{\theta+1})^{j-i}\,t^{j-i}e^{-j\theta t} \qquad (16)$$

Substitute (16) into (13) and do the necessary simplifications, one has

$$\int_{x}^{\infty} t.f(t)dt = \frac{\theta^{2}(p+1)}{(\theta+1)\ln(p)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}[\binom{j}{i}(1-p)^{j+1}\,(\frac{\theta}{\theta+1})^{j-1}\int_{x}^{\infty}\left(t^{j-i+1}+t^{j-i+2}\right)e^{-(j+1)}\,\theta t, \qquad (17)$$

Evaluating the integral $\int_{x}^{\infty}\left(t^{j-i+1}+t^{j-i+2}\right)e^{-(j+1)\theta t}dt$ by using the substitution $u=\theta(j+1)$,

$$\int_{x}^{\infty}\left(t^{j-i+1}+t^{j-i+2}\right)e^{-(j+1)\theta t}dt = \int_{x}^{\infty} t^{j-i+1}+e^{-(j+1)\theta t}\,dt+\int_{x}^{\infty} t^{j-i+2}e^{-(j+1)\theta t}\,dt$$

Then, using

$$A=\int_{x}^{\infty} t^{j-i+1}e^{-(j+1)\theta t}dt$$

and

$$B=\int_{x}^{\infty} t^{j-i+2}e^{-(j+1)\theta t}dt$$

Then,

$$A=\frac{1}{[\theta(j+1)]^{j-i+1}}\int_{\theta(j+1)x}^{\infty} u^{j-i+1}e^{-u}\,du = \frac{r(j-i+2,\theta(j+1)x)}{[\theta(j+1)]^{j-i+1}} \qquad (18)$$

Also,

$$B=\frac{1}{[\theta(j+1)]^{j-i+2}}\int_{\theta(j+1)^{x}}^{\infty}u^{j-i+2}e^{-u}\,du=\frac{r(j-i+3,\theta(j+1)x)}{[\theta(j+1)]^{j-i+2}},\qquad(19)$$

where r(.,.) is the higher incomplete gamma function, and defined by $r(a,b)=\int_{b}^{\infty}x^{a-1}e^{-x}dx.$

Substitute (18) and (19) into (17) and doing the necessary simplifications, gives

$$\int_{x}^{\infty}t.f(t)dt=\frac{(p-1)}{\ln(p)}\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}iCj[\theta(j+1)\Gamma(j-i+2,\theta(j+1)x)+\Gamma(j-i+3,\theta(j+1)x)],\qquad(20)$$

where $\sum_{i=0}^{j}\sum_{j=0}^{\infty}iCj$ is a constant term, and denoted by

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}iCj=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\frac{\binom{j}{i}(1-p)^{j}(1+\theta)^{i-j-1}}{(j+1)^{j-i+3}}\right]$$

Finally, collecting all of the above evaluations the MRL of the Logarithmic-Lindley distribution can be written as

$$m(x)=\frac{(p-1)}{\ln[1-(1-p)(\frac{\theta+1+\theta x}{\theta+1})e^{-\theta x}]}\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\{iCj\ \ [\theta(j+1)\Gamma(j-i+2,\theta(j+1)x)+\Gamma(j-i+3,\theta(j+1)x)]\}-x\qquad(21)$$

4 Statistical Properties

This section investigates the statistical properties of the Logarithmic-Lindley distribution such as the moments, the moment generating function, the quantiles and the median.

4.1 Moments

The r^{th}non-central moment of the Logarithmic-Lindley distribution is given by

$$E(X^r) = \acute{\mu}_r = \frac{\theta^2(p-1)}{(\theta+1)\ln p}\int_0^\infty \frac{(x^r + x^{r+1})e^{-\theta x}}{\left[1-(1-p)\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}\right]}dx, \qquad r = 1,2,\ldots \qquad (22)$$

Using the expansion $(1- \ )^{-1} = \sum_{j=0}^{\infty} z^j \ \ , one\ has$

$$[1-(1-p)\left(1+\frac{\theta x}{(\theta+1)}\right)e^{-\theta x})^{-1} = \sum_{j=0}^{\infty}(1-p)^j(1+\frac{\theta x}{1+\theta})^j e^{-j\theta x} \qquad (23)$$

Similarly, using the expansion $(1+b)^n = \sum_{i=0}^{\infty}\binom{j}{i}b^i$, we can have

$$(1+\frac{\theta x}{1+\theta})^j = \sum_{i=0}^{\infty}\binom{j}{i}\left(\frac{\theta}{1+\theta}\right)^i x^i \qquad (24)$$

Substitute (23) and (24) into (22), gives us

$$\acute{\mu}_r = \frac{\theta^2(p-1)}{(\theta+1)\ln p}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\binom{j}{i}(1-p)^j(\frac{\theta}{1+\theta})^i\int_0^\infty \left(x^{r+i} + x^{r+i+1}\right)e^{-(j+1)\theta x}dt \qquad (25)$$

Finally, evaluating the integral

$$\int_0^\infty \left(x^{r+i} + x^{r+i+1}\right)e^{-(j+1)\theta x}dt$$

and doing the necessary simplifications the r^{th} non-central moment of the Logarithmic- Lindley distribution can be written as

$$\acute{\mu} = \frac{(p-1)}{\theta^r \ln p}$$

Depending on (26), we can conclude the basic statistical properties as follows;

(i) The mean, $\acute{\mu}_1 = \mu$, and the variance, $Var(X)$, of the Logarithmic-Lindley random variable X are, respectively, given by

$$\mu = \frac{(p-1)}{\theta \ln p} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} [\frac{j!\,(1-p)^j [\theta(j+1)+i+2](i+1)}{(1+\theta)^{i+1}(j+1)^{i+3}(j-i)!}], \qquad (27)$$

and

$$Var(X) = \acute{\mu}_2 - \acute{\mu}_1$$

where $\acute{\mu}_2$ is the second non-central moment and given by

$$\acute{\mu}_2 = \frac{(p-1)}{\theta^2 \ln p} \sum_{i=0}^{j} \sum_{j=0}^{\infty} [\frac{j!\,(1-p)^j [\theta(j+1)+i+3](i+2)(i+1)}{(1+\theta)^{i+1}(i+1)^{i+4}(j-i)!}] \qquad (28)$$

(ii) The n^{th} central moments $\mu_n$can be obtained easily from the r^{th} moments through the relation

$$\mu_n = E(x-\mu)^n = \sum_{r=0}^{n} \binom{n}{r} (-\mu)^{n-r} \acute{\mu}_{r.}$$

Then the n^{th} central moments of the Logarithmic-Lindley distribution are given by

$$\mu_x = \frac{(p-1)}{\ln p} \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left[\frac{\binom{j}{i} t^3 (1-p)^j [\theta(j+1)+r+i+1](r+i)}{\theta^3 (1+\theta)^{i+1} (j+1)^{r+i+2}\, r!} \right] \qquad (29)$$

4.3 The Quantiles

Let X be a random variable with cdf associated with (9). Then, the quantile function, x_q, defined by $F\ (xq) = q$ is the root of the equation

$$\left(\frac{\theta+1+\theta x_q}{\theta+1}\right) e^{-\theta x_q} = \left(\frac{1-p^{1-q}}{1-p}\right), 0 < q < 1. \qquad (30)$$

Substituting, $y_{q\,=-1-\theta-\theta x_q,}$we can we can rewrite (30) as

$$y_q e^{yq} = -(\theta+1)e^{-(\theta+1)} \left(\frac{1-p^{1-q}}{1-p}\right), 0 < q < 1 \qquad (31)$$

Hence, the solution of $y_q$is

$$y_{q=W(-(\theta+1)e^{-(\theta+1)}}(\frac{1-p^{1-q}}{1-p})), 0 < q < 1, \qquad (32)$$

where $W(.)$ is the Lambert W function, see Corless et al. (1996) for more details about the properties of the Lambert W function? Inverting (32), one has

$$x_q = -1 - \frac{1}{\theta} - \frac{1}{\theta} W\left(-(\theta+1)e^{-(\theta+1)}\left(\frac{1-p^{1-q}}{1-p}\right)\right), \quad 0 < q < 1. \quad (33)$$

Remark 1 A particular case of (33) at $p \to 1$ gives the quantile function of the Lindley distribution; see Jodra (2010), as

$$x_q = -1 - \frac{1}{\theta} - \frac{1}{\theta} W\left(-(\theta+1)e^{-(\theta+1)}(1-q)\right). \quad (34)$$

When q=0.5 in (33), one can obtain the median of the distribution as

$$x_{0.5} = -1 - \frac{1}{\theta} - \frac{1}{\theta} W\left(\frac{-(\theta+1)e^{-(\theta+1)}}{1+\sqrt{p}}\right), \quad (35)$$

A series expansion for (33) around q = 1 can be obtained as

$$x_q = -1 - \frac{1}{\theta} - \frac{1}{\theta} \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} \quad (36)$$

where $z = -(\theta+1)e^{-(\theta+1)}(\frac{1-p^{1-q}}{1-p})$ These kind of expansions for computing W(.) are widely available, for example, ProductLog[.] in R software.

5 Distribution of Order Statistics

Let $X_1 X_2 \ldots \ldots X_n$ denote n independent random variables from a distribution function $FX(x) = 1 — \overline{F}X\,(x)$ with pdf $fx(x)$, and then the pdf of $X(j)$ (the j order sample arrangement) is given by

$$f(x)_{(j)}(x) = \frac{n!}{(j-1)!\,(n-j)!} f_X(x)[F_X(x)]^{j-1}[1-F_X(x)]^{n-j}, \quad j = 1,2,\ldots \quad (37)$$

Using (8) and (9) into (37), then the pdf of X j according to the Log-L distribution is given by

$$f(x)_{(j)}(x) = \frac{n!}{(j-1)!\,(n-j)!\ln(p)} \left[\frac{(p-1)\left(\frac{\theta^2}{\theta+1}\right)(1+x)e^{-\theta x}}{1-(1-p)\left(\frac{\theta+1+\theta x}{\theta+1)}\right)e^{-\theta x}}\right],$$

$$\times [1 - \frac{[\ln[1-(1-p)(\frac{\theta+1+\theta x}{\theta+1})e^{-\theta x]}}{\ln(p)}]^{j-1},$$

$$\times [\frac{\ln[1-(1-p)(\frac{\theta+1+\theta x}{\theta+1})e^{-\theta x}]}{\ln p}]^{n-1}, \quad . \quad (38)$$

Therefore, the pdf of the largest order statistic $X_{(n)}$ and the smallest order statistic $X_{(1)}$ are, respectively, given by

$$f x_{(n)}(x) = \frac{n}{\ln p}\left[\frac{(p-1)\left(\frac{\theta^2}{\theta+1}\right)(1+x)e^{-\theta x}}{1-(1-p)\left(\frac{\theta+1+\theta x}{\theta+1}\right)e^{-\theta x}}\right]\left[1 - \frac{\ln[1-(1-p)(\frac{\theta+1+\theta x}{\theta+1})e^{-\theta x}}{\ln(p)}\right]^{n-1} \quad (39)$$

and

$$f_{x(1)}(x) = \frac{n}{\ln p}\left[\frac{(p-1)\left(\frac{\theta^2}{\theta+1}\right)(1+x)e^{-\theta x}}{1-(1-p)\left(\frac{\theta+1+\theta x}{\theta+1}\right)e^{-\theta x}}\right]\left[\frac{\ln[1-(1-p)(\frac{\theta+1+\theta x}{\theta+1})e^{-\theta x}}{\ln(p)}\right]^{n-1} \quad (40)$$

6 Measures of Inequality and Uncertainty

In this section, Lorenz and Bonferroni curves are introduced as measures of inequality. Also, Renyi entropy will be mentioned as an important measure of uncertainty.

6.1 Lorenz and Bonferroni curves

Lorenz and Bonferroni curves are the most widely used inequality measures in income and wealth distribution (Kleiber, 2004). In fact, Lorenz and Bonferroni curves are depending *on* the length-biased distribution with pdf $f^*(x)$ defined by

$$f^*(x) = \frac{x.f(x)}{\mu},$$

where $f(x)$ is the pdf of the base distribution with mean μ.

Accordingly, Lorenz and Bonferroni curves, denoted by $L(x)$ and $B(x)$ respectively, are defined by

$$L(x) = \frac{f^*(x)}{\mu}, \qquad and \qquad B(x) = \frac{l(x)}{f(x)}, \quad (41)$$

where, $F^*(x)$ cdf of the length-biased distribution.

Now, we shall derive the expressions of $L(x)$ and $B(x)$ based on $F^*(x)$ and $F^*(x)$ for Log-L distribution. It is easily shown that the pdf of the length biased distribution according to the Log-L distribution can be obtained as follows.

$$f^*(x,\theta,p) = \frac{x.f^*(x,\theta,p)}{\mu}$$

$$= \frac{\theta^2(p-1)}{\mu(\theta+1)\text{In}(p)}\left[\frac{(x+x^2)e^{-\theta x}}{1-(1-p)(\frac{\theta+1+\theta zi)e^{-\theta x}}{\theta+1}}\right] \quad (42)$$

Which cdf, $F^*(x)$, given by

$$F^*(x), = \frac{(p-1)}{\theta\mu\ Inp}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\binom{j}{i}(1-P)^j}{(1+\theta)^{i+1}(j+1)^{i+3}}[\theta(j+1)\gamma(i+2,\theta(j+1)x)+\gamma(i+3,\theta)j+1)x] \quad (43)$$

where γ(.,.) is the lower incomplete gamma function ,defined by

$\gamma(a,b) = \int_a^b u^{a-1}e^{-u}\,du$

It follow from (8) ,(41) and, (43) that L(x) and B(r) are

$$L(x) = \frac{(p-1)}{\theta u^2\ Inp}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\binom{j}{i}(1-P)^j}{(1+\theta)^{i+1}(j+1)^{i+3}}[\theta(j+1)\gamma(i+2,\theta(j+1)x)+\gamma(i+3,\theta)j+1)x] \quad (44)$$

and

$$\text{B(x)} = \frac{(p-1)}{\theta u^2\ [\ In\ p - In\left(1-\left(1-p\left(\frac{\theta+1+\theta xi)}{\theta+1}\right)e^{-\theta xi}\right)\right]}$$

$$\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\binom{j}{i}(1-P)^j}{(1+\theta)^{i+1}(j+1)^{i+3}}[\theta(j+1)\gamma(i+2,\theta(j+1)x)+\gamma(i+3,\theta(j+1)x)] \quad (45)$$

6.2 Renyi Entropy

If X is a random variable having an absolutely continuous cdf $F(x)$ and pdf $f(x)$, then the basic uncertainty measure for distribution F (called the entropy of F) is defined as $[T(x) = E[-ln(f\ (x))]$. Statistical entropy is a probabilistic measure of uncertainty or ignorance about the outcome of a random experiment, and is a measure of a reduction in that uncertainty. Abundant entropy and information indices, among them the Renyi entropy, have been developed and used in various disciplines and contexts. Information theoretic principles and methods have become integral parts of probability and statistics and have been applied in various branches of statistics and related fields.

Renyi entropy is an extension of Shannon entropy. Renyi entropy of the Logarithmic-Lindley distribution is defined to be

$$\gamma_v(f(x,\theta,p) = \frac{\ln\left(\int_0^\infty f^v(x,\theta,p)dt\right)}{1-v} \qquad (46)$$

where $v > 0$ and $v = 1$. Renyi entropy tends to Shannon entropy as v → 1. Now

$$\int_0^\infty f^v(x,\theta,p)dr$$

$$= \left(\frac{\theta^2(p-1)}{(\theta+1)Lnp}\right)^v \int_0^\infty \frac{(1+x)^{ve-v\theta x}}{\left[1-(1-p)\left(\frac{\theta x}{\theta+1}\right)e^{-\theta x}\right]^v} dx \qquad (47)$$

Using the following expansions

$$\left[1-(1-p)\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}\right]^v$$

$$= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\binom{v+j-1}{j}\binom{j}{i}(1-p)^j\left(\frac{\theta}{\theta+1}\right)^i x^i e^{-j\theta x} \qquad (48)$$

and

$$(1+x)^v = \sum_{k=0}^{\infty}\binom{v}{k}x^k \qquad (49)$$

Then, (47) cab be written follows

$$\int_0^\infty f^v(x,\theta,p)dr = \left(\frac{\theta^2(p-1)}{(\theta+1)\ln(p)}\right)^v$$

$$\times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\binom{v}{k}\binom{v+j-1}{j}\binom{j}{i}(1-p)^j\left(\frac{\theta}{\theta+1}\right)\int_0^\infty x^{i+k}\, e$$

$$-(j+v)^{\theta x}dx \qquad (50)$$

Evaluating the integral in (50) using the gamma function. Then, collecting all of the above evaluations and substituting in (46), the Renyi entropy of the Logarithmic-Lindley distribution can be defined as

$$r_v(f(x,\theta,p)) = v\ln\left(\frac{p-1}{\ln(p)}\right) + \ln\left[\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{\binom{v}{k}\binom{v+j+1)}{j}\binom{j}{i}\theta^{2v-k-1}(1-p)^j(i+k)^i}{(\theta+1)^{v+1}(j+v)^{i+k+1}}\right] \quad (51)$$

7 Estimation of the Parameters

In this section we introduce the method of likelihood to estimate the parameters involved and use them to create confidence intervals for the unknown parameters, then gives the equation used to estimate the parameters using the maximum product spacing estimates and the least square estimates techniques.

7.1 Maximum Likelihood Estimation Method

Let $X_1, X_2, ..., X_n$ be a sample size n from Logarithmic-Lindley distribution. Then the likelihood function (1) is given by

$$\prod_{i=1}^{n} fi(x) = \left(\frac{1}{\ln p}\right)^n (p-1)^n\left(\frac{\theta^2}{\theta+1}\right)^n \frac{\prod_{i=1}^{n}(1+xi)e-\theta\sum_{i=1}^{n}x_i}{\prod_{i=1}^{n}\left[1-(1-p)\left(\frac{\theta+1+\theta x_i)}{\theta+1}\right)e^{-\theta x_i}\right]} \quad (52)$$

Hence, the log-likelihood function $\mathcal{L}$= ln 1 becomes

$$\mathcal{L} = -n\ln[\ln p] + n\ln(p-1) + n\ln(\frac{\theta 2}{\theta+1}) + \sum_{i=1}^{n}\ln(1+x_i) - \theta\sum_{i=1}^{n}x_i - \sum_{i=1}^{n}\ln\left[1-(1-p)(\frac{\theta+1+\theta x_i}{\theta+1})e^{-\theta x_i}\right] \quad (53)$$

Therefore, the maximum likelihood estimators (MLEs) of d and p are derived from the derivatives of L. They should satisfy the following equations

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{n(\theta+2)}{\theta(\theta+1)} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{x_i[1-(\theta+1)(\theta+1+\theta x_i)]e^{-\theta x_i}}{(\theta+1)} = 0 \qquad (54)$$

and

$$\frac{\partial \mathcal{L}}{\partial p} = \frac{-n}{p \ln p} + \frac{n}{(p-1)} - \sum_{i=1}^{n} [\frac{(\theta+1+\theta x_i)e^{-\theta x_i}}{1-(1-p)\left(\frac{\theta+1+\theta x_i}{\theta+1}\right)e^{-\theta x_i}}] = 0 \qquad (55)$$

To solve the Equations (54) and (55), it is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function. In order to compute the standard errors and asymptotic confidence intervals we use the usual large sample approximation, in which the MLEs can be treated as being approximately trivariate normal. Hence as n! 1, the asymptotic distribution of the MLE is given by, see Zaindin et al. (2009):

$$\begin{pmatrix} \hat{\theta} \\ \hat{p} \end{pmatrix} \sim \text{Normal} \left[\begin{pmatrix} \theta \\ p \end{pmatrix}, \begin{pmatrix} \hat{v}11 & \hat{v}12 \\ \hat{v}21 & \hat{v}22 \end{pmatrix} \right]$$

where $\hat{v}_{ij}$ $\hat{v}_{ij}$ $|\theta = \theta$ *and*

$$\begin{pmatrix} v11 & v12 \\ v21 & v22 \end{pmatrix} = \begin{pmatrix} A11 & A12 \\ A21 & A22 \end{pmatrix}$$

is the approximate variance-covariance matrix with its elements obtained from

$$A_{11} = \frac{\partial^2 \mathcal{L}}{\partial \theta^2}, \qquad A_{12} = \frac{\partial^2 \mathcal{L}}{\partial \theta \partial p^{\mathcal{L}}} \quad \text{and} \quad A_{22} = \frac{\partial^2 \mathcal{L}}{\partial p^2}$$

By solving this inverse dispersion matrix, these solutions will yield the asymptotic variances and covariances of these MLEs for θ and p.

Approximate 100(1 - a)% confidence intervals for 9 and p can be determined as

$$\hat{\theta} \pm Z_{\frac{\propto}{2}} \sqrt{\hat{v}11} \quad \text{and} \quad \hat{p} \mp Z_{\frac{\propto}{2}} \sqrt{\hat{v}22}.$$

where $Z_{\frac{\propto}{2}}$ is the upper ath percentile of the standard normal distribution.

7.2 Maximum Product Spacing Estimates

The maximum product spacing (MPS) method has been proposed by Cheng and Amin (1983). This method is based on an idea that the differences (spacings) of the consecutive points should be identically distributed. The geometric mean of the differences is given as

$$\mathrm{GM} = \sqrt[n+1]{\prod_{i=1}^{n+1} AD_i} \qquad (56)$$

where is difference D_i is defined as

$$D_i = \int_{x_{(i-1)}}^{x_{(i)}} f(x,\theta,p)dx; i = 1,2,\ldots\ldots,n+1 \quad (57)$$

Where, $F(x_{(0)},\theta,p)$ =0 and $F(x(n+1),\theta,p)$ = 1. The MPS estimators θ_{PS} and p_{PS} of θ and p are obtained by maximizing the geometric mean (GM) of the differences. substituting pdf of Logarithmic-Lindley distribution in (57) and taking logarithm of the above expression, one can have.

$$\log GM = \frac{1}{n+1}\sum_{i=1}^{n+1} \log \left[F\left(x_{(i)},\theta,p\right) - F\left(x_{(i-1)},\theta,p\right)\right] \quad (58)$$

The MPS **estimator** θ_{PS} and p_{PS} of θ and p can be obtained as the simultaneous solution of the following non- linear equations.

$$\frac{\partial \log GM}{\partial \theta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{\acute{F}\theta(x_{(i)},\theta,p - \acute{F}_\theta(x_{(i-1),\theta,p}}{\acute{F}\theta(x_{(i)},\theta,p - F_\theta(x_{(i-1),\theta,p}}\right] = 0,$$

$$\frac{\partial \log GM}{\partial p} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{\acute{F}p(x_{(i)},\theta,p - \acute{F}_p(x_{(i-1),\theta,p}}{\acute{F}p(x_{(i)},\theta,p - F_p(x_{(i-1),\theta,p}}\right] = 0,$$

7.3 Least Square Estimates

Let $x_{(1)}$, $x_{(2)}$,..., $x_{(n)}$ be the ordered sample of size n drawn from the Logarithmic-Lindley population pdf. Then, the expectation of the empirical cumulative distribution function is defined as

$$E\left[F\left(X_{(i)}\right)\right] = \frac{i}{n+1}; i = 1,2,\ldots\ldots\ldots n \quad (59)$$

The least square estimates (LSEs) θ_{LS} and p_{LS} of θ and p are obtained by minimizing

$$Z(\theta,p)=\sum_{i=1}^{n}\left(F(x_i),\theta,p)-\frac{i}{n+1}\right)^2$$

Therefore, θ_{LS} and p_{LS} of θ and p can be obtained as the solution of the following system of equations

$$\frac{\partial Z(\theta,p)}{\partial\theta}=\sum_{i=1}^{n}F\acute{\theta}\left(X_{(i)}\theta,p\right)\left(F\left(X_{(i)}\theta,p\right)-\frac{i}{n=1}\right)=0,$$

and

$$\frac{\partial Z(\theta,p)}{\partial p}=\sum_{i=1}^{n}F\acute{p}\left(X_{(i)}\theta,p\right)\left(F\left(X_{(i)}\theta,p\right)-\frac{i}{n=1}\right)=0,$$

Theses non-linear equations can be routinely solved by using Newton's method or fixed point iteration techniques. The subroutines to solve nonlinear optimization problem are available in R software namely optim (), n/m () and bfrm/e () etc., see Team (2012). We used n/m () package for optimizing Equations (53) and (58).

8 Applications

In this section, we use two real data sets to show that the Logarithmic-Lindley distribution can be better than one based on the Lindley distribution and other non-nested models such as Lindley-exponential by Bhatti (2014), new generalized Lindley (NGL) distribution by Elbatal et al. (2013), Weibull distribution by Weibull (1939) and weighted Lindley distribution by Ghitany et al. (2011).

8.1 Data Set 1

The first data set represents an uncensored data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang (2003). In order to compare the six distribution models, we consider criteria like A/C (Akaike information criterion), A/CC (corrected Akaike information criterion) and B/C (Bayesian information criterion) for the data set. The better distribution corresponds to smaller A/C, A/CC and B/C values:

$AIC = 2k - 2i,$

$$AICC = AIC = \frac{2k(k+1)}{n-k-1}\,and\;\;BIC = 21 + K * \log_{(n)}$$

where k is the number of parameters in the statistical model, n the sample size and r is the maximized value of the log-likelihood function under the considered model.

The LR test statistic to test the hypotheses Ho : p = 1 versus H_1 : p = 1 is ! = 14.032 > 7.815 = xi 05, so we reject the null hypothesis. Table 1 shows parameter MLEs to each one of the six fitted distributions for data set with 95% confidence interval, while Table 3 represents the values of - log(L), AIC, BIC and AICC.

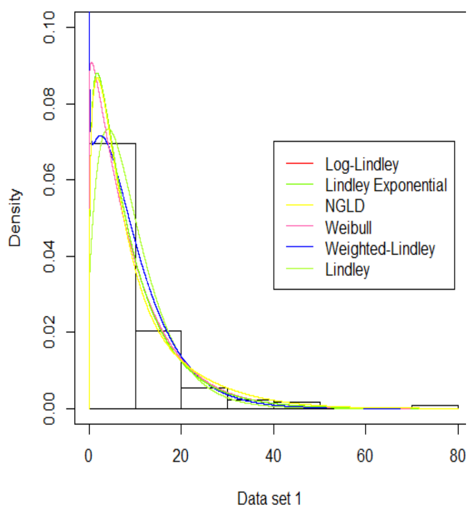
The values in Table 2, indicate that the Logarithmic-Lindley is a strong competitor to other distributions used here for fitting data set. A density plot compares the fitted densities of the models with the empirical histogram of the observed data (Fig. 3). The fitted density for the Logarithmic-Lindley model is closer to the empirical histogram than the fits of the other models.

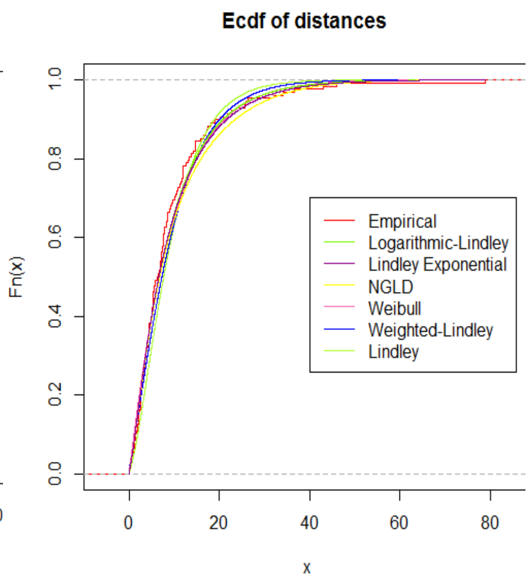
TABLE 1. Maximum likelihood estimates with 95% CI for data set1

Model	Parameter Es.	St. Err.	95% CI
Logarithmic-Lindley	0.1238	0.0186	[0.0872, 0.1604]
	0.0979	0.0472	[0.0053, 0.1906]
Lindley- Exponential	0.1093	0.0137	[0.0824, 0.1363]
	1.5687	0.1638	[1.2476, 1.8898]
NGL	0.1827	0.0355	[0.1130, 0.2525]
	4.6807	1.3080	[2.1169, 7.2445]
	1.3243	0.1718	[0.9874, 1.6611]
Weibull	1.0478	0.0676	[0.9153,1.1803]
	0.1045	0.0093	[0.0862, 0.1229]
Weighted Lindley	0.1594	0.0172	[0.1257, 0.1931]
	0.6827	0.1115	[0.4640, 0.9014]
Lindley	0.1960	0.0123	[0.1718, 0.2202]

TABLE 2. - log L, AIC, AICC, BIC, KS statistics values under considered models based on data set 1.

Model	- log L	AIC	AICC	BIC	KS
Logarithmic-Lindley	411.7701	827.5403	827.6363	833.2443	0.0619
Lindley-Exponential	412.0493	828.0985	828.1945	833.8026	0.0621
NGL	412.7503	831.5006	831.6942	840.0567	0.0740
Weibull	414.0869	832.1738	832.2698	837.8778	0.0701
Weighted Lindley	416.4422	836.8845	836.9805	842.5885	0.0925





(a) (b)

FIGURE 3. (a) Estimated densities of data set 1. (b) Empirical, Logarithmic-Lindley, Lindley, Weibull, Lindley-exponential, NGL, and Weighted Lindley cdf of data set 1.

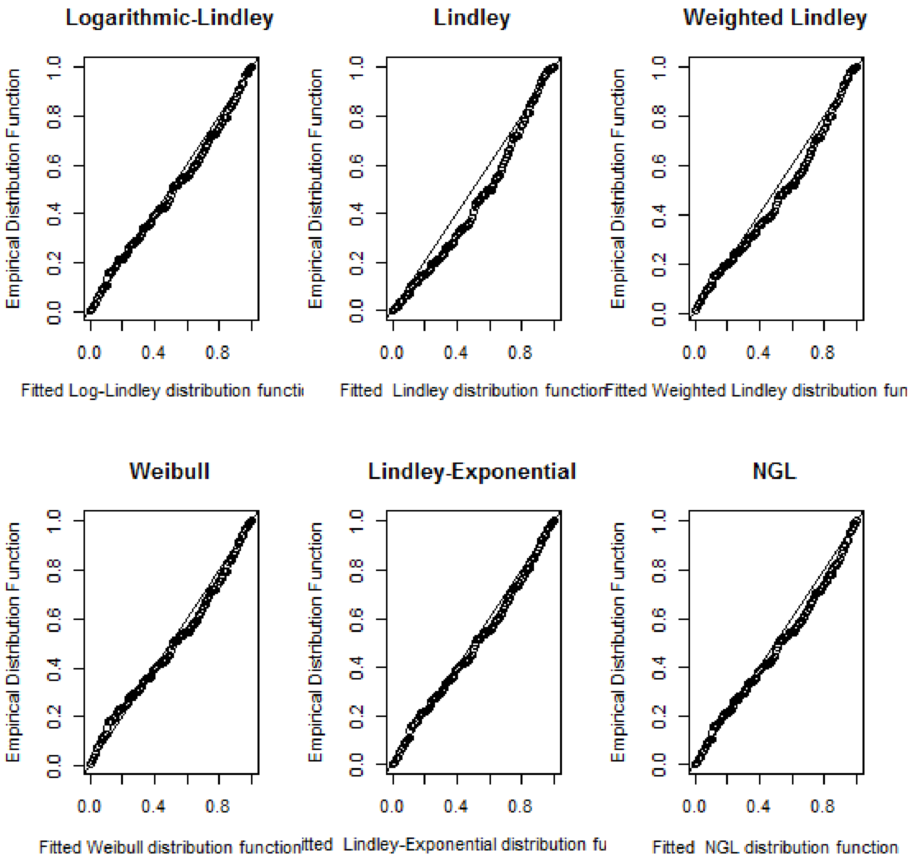


FIGURE 4. Probability plots for the …ts of the Logarithmic-Lindley, Lindley, Weibull, Lindley-exponential, weighted Lindley and NGL distributions of data set 1.

8.2 Data Set 2

The data represents 46 repair times (in hours) for an airborne communication transceiver and available in Chhikara and Folks (1977).

The LR test statistic to test the hypotheses H_0: p = 1 versus H_1 : p = 1 is ! = 8.3546 > 3.841 = x1 $_{-0:05}$, so we reject the null hypothesis.

Table 3 shows parameter MLEs to each one of the six fitted distributions for data set 2 with 95% confidence interval, while Table 4 represents the values of - log(L), AIC, BIC and AICC.

TABLE 3. Maximum likelihood estimates with 95% CI for data set2

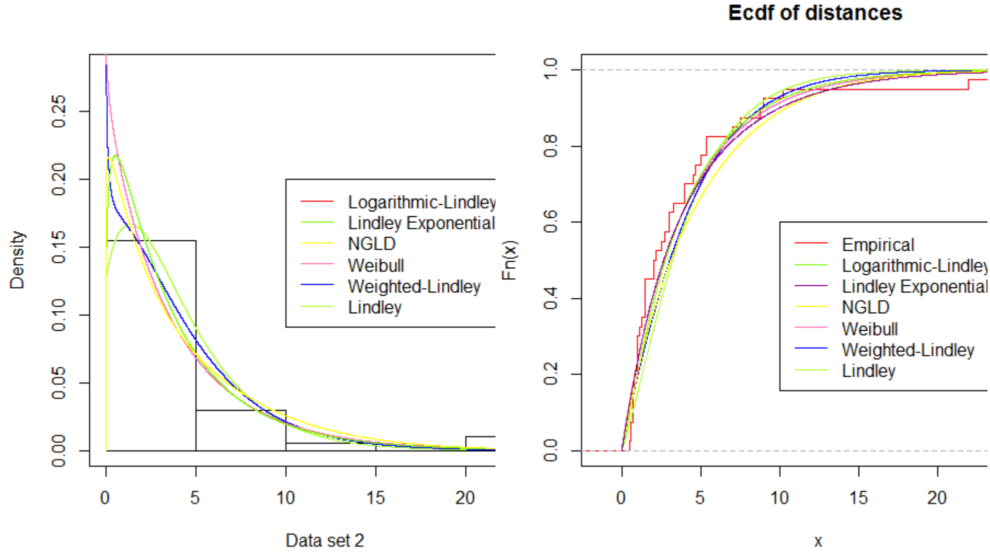
Model	Parameter Es.	St. Err.	95% CI
Logarithmic-Lindley	0.2473	0.0640	[0.0872, 0.1604]
	0.0694	0.0313	[0.0080, 0.1307]
Lindley- Exponential	0.242675	0.05649785	[0.1319393, 0.3534108]
	1.460206	0.2715473	[0.9279731, 1.992438]
NGL	0.3552648	0.1721713	[0.01780907, 0.6927206]
	3.089682	3.338087	[2.02514, 9.632333]
	1.058484	0.2930719	[0.4840632, 7.601135]
Weibull	0.960359	0.06812665	[0.8268308, 1.093887]
	0.2546432	0.1467211	[0.02586985, 0.4834164]
Weighted Lindley	0.3551453	0.06812665	[0.221617,0.4886735]
	0.7471963	0.1867211	[0.381223,1.11317]
Lindley	0.4242097	0.04852818	[0.3290945, 0.519325]

The values in Table 4, indicate that the Logarithmic-Lindley is a strong competitor to other distributions used here for fitting data set.

A density plot compares the fitted densities of the models with the empirical histogram of the observed data (Fig. 5). The fitted density for the Logarithmic-Lindley model is closer to the empirical histogram than the fits of the Lindley distribution and other non-nested models.

TABLE 4. - log L, AIC, AICC, BIC, KS statistics values under considered models based on data set 2.

Model	- log L	AIC	AICC	BIC	KS
Logarithmic-Lindley	94.39982	192.7996	193.124	196.1774	0.1212
Lindley-Exponential	94.614	193.2284	193.5527	196.6062	01488
NGL	96.13662	198.2732	198.9399	203.3399	0.1621635
Weibull	95.51136	195.0227	195.0347	198.4005	0.1224559
Weighted Lindley	98.04943	200.0989	200.4232	203.4766	0.1722701
Lindley	98.79132	199.5826	199.6879	201.2715	0.2156951



(a) (b)

FIGURE 5. (a) Estimated densities of data set 1.
(b) Empirical, Logarithmic-Lindley, Lindley,
Weibull, Lindley-exponential, NGL, and
weighted Lindley cdf's of data set 2.

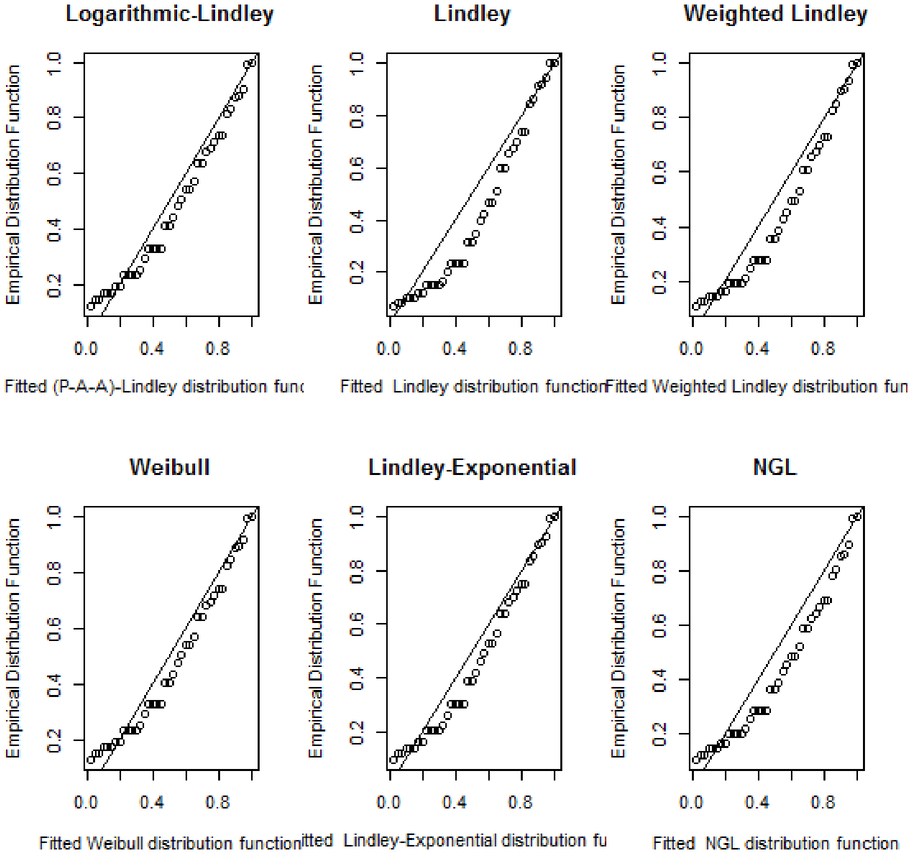


FIGURE 6. Probability plots for the Logarithmic-Lindley, Lindley, Weibull, Lindley-exponential, weighted Lindley and NGL distributions of data set 2.

9 Conclusion

Here, we propose a new model, the so-called the Logarithmic-Lindley distribution which extends the Lindley distribution in the analysis of data with real support. An obvious reason for generalizing a standard distribution is because the generalized form provides larger flexibility in modeling real data. We derive expansions for the moments, moment generating function, hazard rate function, reversed hazard rate function, cumulative hazard rate function, mean residual lifetime distribution, quantiles, Lorenz curves, Bonferroni curves and Renyi entropy. The distribution of order statistics is presented

according to the proposed model. The estimation of parameters is approached by the methods of maximum likelihood, maximum product spacing's and least squares, also the information matrix is derived. We consider the likelihood ratio statistic to compare the model with its baseline model. Two applications of the Logarithmic-Lindley distribution to real data show that the new distribution can be used quite effectively to provide better fits than the Lindley distribution and other non-nested models such as Lindley-exponential, weighted Lindley, Weibull and new generalized Lindley distributions. Finally, we followed our work by a simulation algorithms and study.

Conflicts of Interest: The authors declare that there is no conflict of interest regarding the publication of this article.

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